Power Series **Power Series Day 1**

Most functions can be represented as a <u>Power Series</u>. In other words, any function can be represented by an infinite sum of power functions.

$$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

Let's look at the function $f(x)=\dfrac{1}{1-x}$. This can be rewritten as a power series.

1 – Long Division

$$\begin{array}{r|l} & 1+x+x^2 \\ \hline 1-x & 1+0x+0x^2+0x^3+0x^4+\ldots \\ & -1+x \\ \hline & \quad +\cancel{x}+0x^2+0x^3+0x^4+\ldots \\ & \quad -\cancel{x}+x^2 \\ \hline & \qquad \cancel{x^2}+0x^3+0x^4+\ldots \\ & \qquad -\cancel{x^2}+x^3 \\ \hline & \qquad\quad x^3+0x^4+\ldots \end{array}$$

$$f(x)=\frac{1}{1-x}=1+x+x^2+x^3+x^4+\ldots=\sum_{n=0}^{\infty}(x)^n$$

2 – Geometric Series $\sum_{n=0}^{\infty}(x)^n$

converges

$|x|<1$

$x<1$ and $x>-1$

$-1<x<1$ Inequality Notation

$(-1,1)$ Interval Notation

$\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n \to$ Last unit

$\sum_{n=0}^{\infty} x^n \to$ converges $(-1,1)$

3 – Graphically: <u>https://www.desmos.com/calculator/aozwuxmec1</u>

So, the function $\frac{1}{1-x} = 1+x+x^2+x^3+x^4+x^5+\ldots = \sum_{n=0}^{\infty} x^n$ memorize

Rewrite each function as an infinite power series. Determine the values for wich the series is valid (interval of convergence).

Know: $f(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$

Ex. 1 $f(x)=\frac{1}{1-2x}$

$f(x)=\frac{1}{1-(2x)}=\sum_{n=0}^{\infty}(2x)^n=\sum_{n=0}^{\infty}2^n x^n$

I.O.C. $|2x|<1$

$2x<1$ & $2x>-1$

$x<\frac{1}{2}$ & $x>-\frac{1}{2}$ $\boxed{(-\frac{1}{2},\frac{1}{2})}$

Ex. 2 $f(x)=\frac{1}{1+x}$

Know: $\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$

$-3, -2, -1, 0, 1, 2, 3$
O E O E O E O

$f(x)=\frac{1}{1-(-x)}$

$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n=1-x+x^2-x^3+x^4$

memorize

I.O.C $|-x|<1$

$x<1$ & $x>-1$

$\boxed{(-1,1)}$

Ex. 3 $f(x)=\frac{1}{2+x^2}$

Know: $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$

$f(x)=\frac{1}{2[1+\frac{x^2}{2}]}$

$\frac{1}{2+x^2}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^2}{2}\right)^n\cdot\frac{1}{2}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n}\cdot\frac{1}{2^1}$

$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{n+1}}$

I.O.C $|\frac{x^2}{2}|<1$

$\frac{x^2}{2}<1$ & $\frac{x^2}{2}>-1$

$x^2<2$ $\cancel{\sqrt{x^2}>\sqrt{-2}}$ garbage

$x^2<2$

$x<\sqrt{2}$ & $x>-\sqrt{2}$

$\boxed{(-\sqrt{2},\sqrt{2})}$

Ex. 4 $f(x)=\frac{2x}{1+x^2}$

Know: $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$

$f(x)=2x\cdot\left(\frac{1}{1+x^2}\right)$

$\frac{2x}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n(x^2)^n\cdot 2x^1=\sum_{n=0}^{\infty}(-1)^n\cdot 2\cdot x^{2n+1}$

I.O.C $|x^2|<1$

$\sqrt{x^2}<\sqrt{1}$ & $\cancel{\sqrt{x^2}>\sqrt{-1}}$ garbage

$x<1$ & $x>-1$ $\boxed{(-1,1)}$

Practice: Rewrite each function as a power series. Find the interval of convergence.

1. $f(x) = \frac{1}{2-x}$

Know: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$f(x) = \frac{1}{2(1-\frac{x}{2})}$

$\frac{1}{2-x} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n \cdot \frac{1}{2} = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$

I.O.C $\left|\frac{x}{2}\right| < 1$

$\frac{x}{2} < 1$ & $\frac{x}{2} > -1$ $\quad x < 2$ & $x > -2$ $\quad \boxed{(-2, 2)}$

2. $f(x) = \frac{x}{x^3 - 1}$

$f(x) = x\left(\frac{1}{-1+x^3}\right)$

$f(x) = \frac{x}{-1}\left(\frac{1}{1-x^3}\right)$

$f(x) = -x\left(\frac{1}{1-x^3}\right)$

Know: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$\frac{x}{x^3-1} = \sum_{n=0}^{\infty} (x^3)^n(-x) = \sum_{n=0}^{\infty} -1 \cdot x^{3n+1}$

I.O.C $|x^3| < 1$

$x^3 < 1$ & $x^3 > -1$ $\quad \boxed{(-1, 1)}$

3. $f(x) = \frac{1}{x}$

$f(x) = \frac{1}{1+(x-1)}$ $\qquad f(x) = \frac{1}{1-(1-x)}$ → Both Work

Know: $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

$\frac{1}{1+(x-1)} = \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$

I.O.C

$|x-1| < 1$

$x-1 < 1$ & $x-1 > -1$

$x < 2$ & $x > 0$

$\boxed{(0, 2)}$