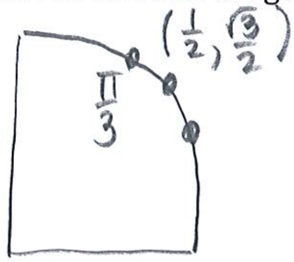
Notes: Finding the Equation of Tangent Lines, Vertical, and Horizontal Tangents of Polar

P, P, & V Day 9

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} sin\theta + rcos\theta}{\frac{dr}{d\theta} cos\theta - rsin\theta} = \frac{f'(\theta)sin\theta + f(\theta)cos\theta}{f'(\theta)cos\theta - f(\theta)sin\theta}$$

Example One: Find the equation of the line tangent to $r = 2sin\theta$ in rectangular form given $\theta = \frac{\pi}{3}$.



$$\frac{d}{d\theta}\left[r = 2\sin\theta\right]$$

$$\frac{dR}{d\theta} = 2\cos\theta$$

$$\frac{dy}{dx} = \frac{(2\cos\theta)(\sin\theta) + (2\sin\theta)\cos\theta}{(2\cos\theta)(\cos\theta) - (2\sin\theta)\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\frac{\pi}{3}} = \frac{2\cos\frac{\pi}{3}\sin\frac{\pi}{3} + 2\sin\frac{\pi}{3}\cos\frac{\pi}{3}}{2\cos\frac{\pi}{3}\cos\frac{\pi}{3} - 2\sin\frac{\pi}{3}\sin\frac{\pi}{3}}$$

$$= \frac{\cancel{2}\left(\cancel{\frac{1}{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \cancel{2}\left(\cancel{\frac{\sqrt{3}}{2}}\right)\left(\frac{1}{2}\right)}{\cancel{2}\left(\cancel{\frac{1}{2}}\right)\left(\frac{1}{2}\right) - \cancel{2}\left(\cancel{\frac{\sqrt{3}}{2}}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}}{\frac{1}{2} - \frac{3}{2}} = \frac{\frac{\cancel{2}\sqrt{3}}{\cancel{2}}}{-\frac{\cancel{2}}{\cancel{2}}} = \boxed{-\sqrt{3}}$$

Tangent Line

$\boxed{1.}$ Point: $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

$\boxed{2}$ Slope: $m = -\sqrt{3}$

$$\boxed{y - \frac{3}{2} = -\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)}$$

Polar $\longrightarrow$ rectangular

$r = 2\sin\theta$, $\theta = \frac{\pi}{3}$ $\Rightarrow$ $x = r\cos\theta$, $y = r\sin\theta$

$$x = r\cos\theta$$
$$x = (2\sin\theta)(\cos\theta)$$
$$x = 2\sin\frac{\pi}{3}\cos\frac{\pi}{3}$$
$$x = \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$$

$$y = r\sin\theta$$
$$y = (2\sin\theta)\sin\theta$$
$$y = 2\sin\frac{\pi}{3}\sin\frac{\pi}{3}$$
$$y = \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2}$$

$\frac{d}{d\theta}[r = 2 - 2\cos\theta]$ $r' = 2\sin\theta$

Example Two: Find the points in rectangular where $r = 2(1 - cos\theta)$ has vertical and horizontal tangents.

$$\frac{dy}{dx} = \frac{(2\sin\theta)\sin\theta + (2-2\cos\theta)\cos\theta}{(2\sin\theta)\cos\theta - (2-2\cos\theta)\sin\theta}$$

$\boxed{\text{Horizontal Tangent}}$ (top = 0) ☺

$2(\sin^2\theta) + 2\cos\theta - 2\cos^2\theta = 0$

$2(1-\cos^2\theta) + 2\cos\theta - 2\cos^2\theta = 0$

$2 - 2\cos^2\theta + 2\cos\theta - 2\cos^2\theta = 0$

$$\frac{-4\cos^2\theta + 2\cos\theta + 2 = 0}{-2}$$

$2\cos^2\theta - \cos\theta - 1 = 0$

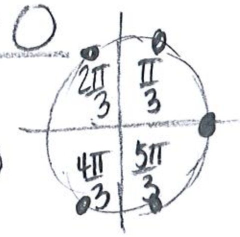
$2x^2 - x - 1 = 0$

$(2x+1)(x-1) = 0$

$2\cos\theta + 1 = 0$ $\cos\theta - 1 = 0$

$\cos\theta = -\frac{1}{2}$ $\cos\theta = 1$

$\theta = \frac{2\pi}{3}$ & $\frac{4\pi}{3}$ $\theta = 0$



Polar Coordinates

(r, θ) $r = 2 - 2\cos\theta$

$(3, \frac{2\pi}{3})$, $(3, \frac{4\pi}{3})$ $\rangle$ $r = 2 - 2(-\frac{1}{2})$

$\cancel{(0, 0)} \rightarrow r = 2 - 2(1)$

$\boxed{\text{Vertical tangents}}$ (bottom = 0)

$2\sin\theta\cos\theta - 2\sin\theta + 2\sin\theta\cos\theta = 0$

$4\sin\theta\cos\theta - 2\sin\theta = 0$

$2\sin\theta(2\cos\theta - 1) = 0$

$2\sin\theta = 0$ $2\cos\theta - 1 = 0$

$\sin\theta = 0$ $\cos\theta = \frac{1}{2}$

$\theta = 0, \pi$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

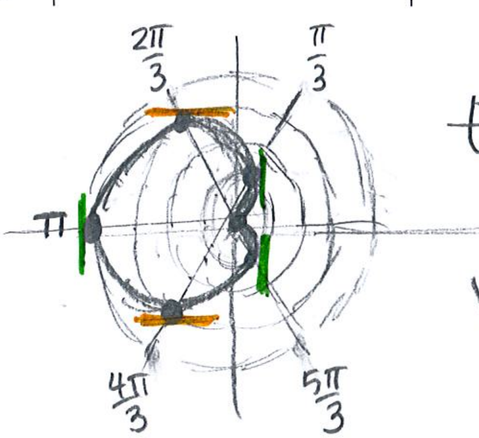
$\cancel{(0, 0)}$ $(4, \pi)$

$r = 2 - 2\cos(0)$

$r = 2 - 2\cos\pi$

$(1, \frac{\pi}{3})$, $(1, \frac{5\pi}{3})$ $\rangle$ $r = 2 - 2\cos\theta$, $r = 2 - 2(\frac{1}{2})$

(r, θ)	$x = r\cos\theta$	$y = R\sin\theta$	(x, y)
$\cancel{(0, 0)}$	$x = 0\cos(0) = 0$	$y = 0\sin(0) = 0$	$(0, 0)$
$(3, \frac{2\pi}{3})$	$x = 3\cos(\frac{2\pi}{3})$ $3(-\frac{1}{2})$	$y = 3\sin(\frac{2\pi}{3})$ $3(\sqrt{3}/2)$	$(-\frac{3}{2}, \frac{3\sqrt{3}}{2})$
$(3, \frac{4\pi}{3})$	$x = 3\cos(\frac{4\pi}{3})$ $3(-\frac{1}{2})$	$y = 3\sin(4\pi/3)$ $3(-\sqrt{3}/2)$	$(-\frac{3}{2}, -\frac{3\sqrt{3}}{2})$
$(4, \pi)$	$x = 4\cos\pi$ $4(-1)$	$y = 4\sin\pi$ $4(0)$	$(-4, 0)$
$(1, \frac{\pi}{3})$	$x = 1\cos\frac{\pi}{3}$	$y = 1\sin\frac{\pi}{3}$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
$(1, \frac{5\pi}{3})$	$x = 1\cos\frac{5\pi}{3}$	$y = 1\sin\frac{5\pi}{3}$	$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$



$\theta = 0$ is not a horizontal tangent or vertical tangent.

$\theta = 0$ is a cusp