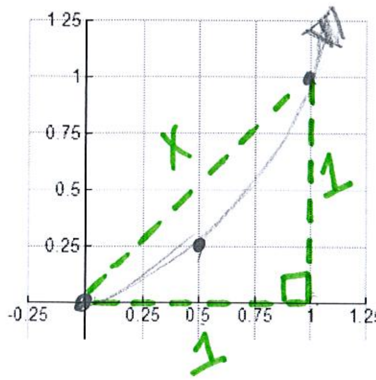
Notes: Arc Length Rectangular & Parametric

Rectangular: Arc Length over $[a, b]$: $Arc\ Length = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Example One: Using your calculator determine the arc length of $y = x^2$ from $[0,1]$.



Approx length:

$1^2 + 1^2 = x^2$

$1 + 1 = x^2$

$\sqrt{2} = \sqrt{x^2}$

$x = \sqrt{2} \approx 1.414$

$y' = 2x$

$\int_0^1 \sqrt{1 + [2x]^2}\, dx$

$\int_0^1 \sqrt{1 + 4x^2}\, dx$

1.479

Example Two: Calculate by hand the arc length of the graph of $f(x) = 2x - 4$ over $[1,3]$.

$f'(x) = 2$

$\text{Arc length} = \int_1^3 \sqrt{1 + [f'(x)]^2}\, dx$

$= \int_1^3 \sqrt{1 + [2]^2}$

$= \int_1^3 \sqrt{5} = \sqrt{5}\,x \Big|_1^3 = 3\sqrt{5} - 1\sqrt{5} = \boxed{2\sqrt{5}}$

Example Three: Determine the length of $f(x) = \ln(secx)$ between $0 \le x \le \frac{\pi}{4}$. (By Hand)

$f'(x) = \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \tan x = \tan x$

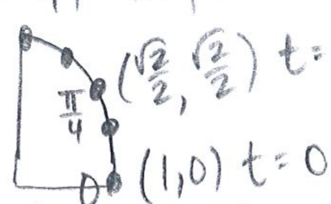
$\text{Arc length} = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x}\, dx$

$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx$

$\int_0^{\pi/4} \sec x\, dx$

$\ln|\sec x + \tan x| \Big]_0^{\pi/4}$

$\ln|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}| - \ln|\sec 0 + \tan(0)|$



$\ln|\cancel{\tfrac{2\sqrt{2}}{2\sqrt{2}}} + 1| - \ln|\cancel{1+0}|$ → 0

$\boxed{\ln|\sqrt{2} + 1|}$

Example Four: Determine the length of the curve of $y = \frac{2}{3}(x-1)^{\frac{3}{2}}$ on $[1,4]$. By hand.

$$y' = \cancel{\tfrac{2}{3}} \cdot \cancel{\tfrac{3}{2}}(x-1)^{1/2}(1) = \sqrt{x-1}$$

$$\text{Arc length} = \int_1^4 \sqrt{1+(\sqrt{x-1})^2}\,dx$$

$$\int_1^4 \sqrt{\cancel{1} + x - \cancel{1}}\,dx$$

$$\int_1^4 x^{1/2}\,dx$$

$$\frac{2}{3}x^{3/2}\Big|_1^4 = \frac{2}{3}\left[\underset{2^3}{(\sqrt{4})^3} - \underset{1^3}{(\sqrt{1})^3}\right] = \frac{2}{3}(7) = \boxed{\frac{14}{3}}$$

Parametric: $Arc\ Length\ over\ [a,b] = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$

Find the arc length over the given interval:

Example Five:

$C(t) = (4t-3, -3t+1) \qquad 0 \le t \le 5$

$$x'(t) = 4 \quad y'(t) = -3$$

$$\text{Arc length} = \int_0^5 \sqrt{(4)^2 + (-3)^2}\,dt$$

$$\int_0^5 5\,dt$$

$$5t\Big|_0^5$$

$$5(5) - 5(0)$$

$$\boxed{25}$$

Example Six:

$C(t) = \left(\frac{5}{2}t^2, 4t^3 - 2\right)$ $\quad 0 \le t \le 2$

$x'(t) = \frac{5}{2}(2t) = 5t \quad y'(t) = 12t^2$

$\int_0^2 \sqrt{(5t)^2 + (12t^2)^2}\, dt$

$\int_0^2 \sqrt{25t^2 + 144t^4}\, dt$

$\int_0^2 \sqrt{t^2(25 + 144t^2)}\, dt$

$\int_0^2 t\sqrt{25 + 144t^2}\, dt$

$\frac{1}{288}\int_0^2 288t\sqrt{25+144t^2}\, dt$

$u = 25 + 144t^2$

$du = 288t$

$\frac{1}{288}\int_{25}^{601} u^{1/2}\, du$

$u(0) = 25 + 144(0)^2$

$u(0) = 25$

$u(2) = 25 + 144(2)^2$

$u(2) = 601$

$\frac{1}{288}\frac{2}{3}u^{3/2}\Big|_{25}^{601}$

$\frac{1}{432}\left[(\sqrt{601})^3 - (\sqrt{25})^3\right.$

$\boxed{33.816}$

Remember: These are on Notecards ☺

$sin^2x + cos^2x = 1$

$tan^2x + 1 = sec^2x$ ☺

$1 + cot^2x = csc^2x$

$\sin(2x) = 2sinxcosx$

$\cos(2x) = cos^2x - sin^2x$

$= 2cos^2x - 1$

$= 1 - 2sin^2x$

Example Seven:

$C(t) = (\sin(5t), \cos(5t))$ $\quad 0 \le t \le \pi$

$x'(t) = 5\cos(5t) \quad y'(t) = -5\sin(5t)$

$\int_0^\pi \sqrt{(5\cos(5t))^2 + (-5\sin 5t)^2}\, dt$

$\int_0^\pi \sqrt{25\cos^2(5t) + 25\sin^2(5t)}\, dt$

$\int_0^\pi \sqrt{25(\cos^2 5t + \sin^2 5t)}\, dt$

$\int_0^\pi \sqrt{25(1)}\, dt = \int_0^\pi 5\, dt = 5t\Big|_0^\pi = 5\pi - 5(0) = \boxed{5\pi}$