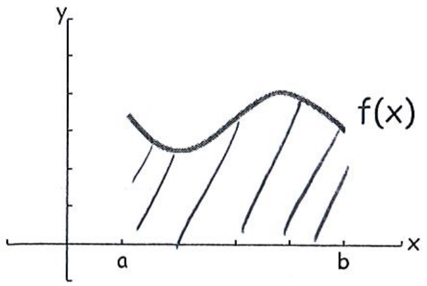
Notes: Finding Area of Bounded Functions in Parametric Form

Before we can find area of parametric equations, lets take a look back of how we find the area bounded between a function and the x-axis ☺



$$\int_a^b f(x)\,dx$$

$$\int_{Lt}^{Rt} y(t)\cdot\frac{dx}{dt}\cdot dt$$

$$\int_{Lt}^{Rt} y(t)\cdot x'(t)\,dt$$

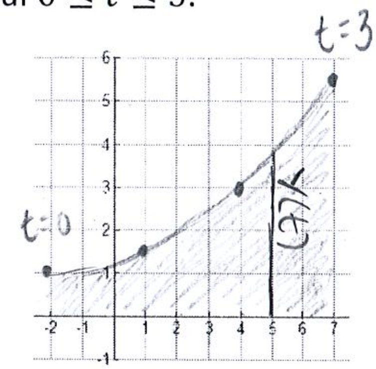
$$Area = \int_{t_0}^{t_1} y(t)x'(t)dt =$$

Example One: Find the area of the region bounded by the parametric curve $x(t) = 3t - 2$, $y(t) = \frac{1}{2}t^2 + 1$ on the interval $0 \le t \le 3$.

t	$x(t)$ $3t-2$	$y(t)$ $\frac{1}{2}t^2+1$
0	-2	1
1	1	1.5
2	4	3
3	7	5.5

√ on calculator $\boxed{2nd}\boxed{7}$

$\int(3 \div 2t \wedge 2 + 3, t, 0, 3)$



$$\int_0^3 \left(\tfrac{1}{2}t^2+1\right)3\,dt$$

$$\int_0^3 \tfrac{3}{2}t^2 + 3\,dt$$

$$\cancel{\tfrac{3}{2}}\tfrac{t^3}{\cancel{3}} + 3t \Big|_0^3$$

$$\frac{3^3}{2} + 3(3) - 0 - 0$$

$$\frac{27}{2} + \cancel{9}\ \frac{18}{2} = \frac{45}{2}$$

Example Two: Sketch the graph of $C(t) = (ln t + 1, 3 - t)$ for $1 \le t \le 3$ and compute the area.

t	$\ln t + 1$	$3 - t$
1	1	2
1.5	1.4	1.5
2	1.7	1
2.5	1.9	.5
3	2.1	0

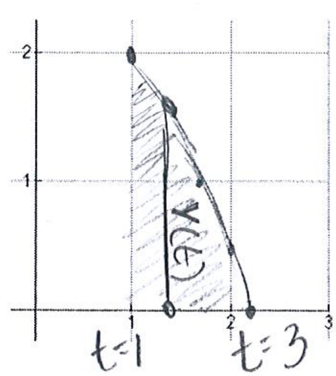
√ on calculator

$\boxed{TI-89}$

$\boxed{2nd}\boxed{7}$ $\int((3-t)*(1\div t), t, 1, 3)$

$\boxed{TI-84}$ $\boxed{math}$ scroll down to fnint

$\int_{\boxed{1}}^{\boxed{3}} \boxed{(3-x)(1\div x)}\,dx$



$$\int_1^3 (3-t)\left(\tfrac{1}{t}\right)dt$$

$$\int_1^3 3\left(\tfrac{1}{t}\right) - t\left(\tfrac{1}{t}\right)dt$$

$$\int_1^3 3\left(\tfrac{1}{t}\right) - 1\,dt$$

$$3\ln t - t \Big|_1^3$$

$$3\ln 3 - 3 - \cancel{3\ln 1}^{\,0} + 1$$

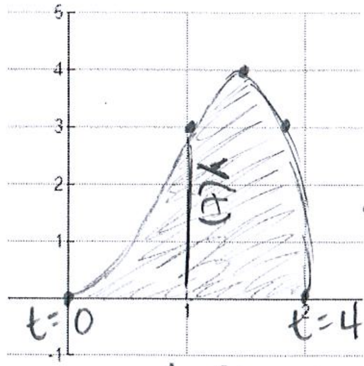
$$3\ln 3 - 2$$

$$\approx 1.296$$

Example Three: Find the area of the region bounded by the parametric curve $x(t) = \sqrt{t}$, $y(t) = 4t - t^2$ and the x-axis.

t	$x(t)$ $\sqrt{t}$	$y(t)$ $4t-t^2$
0	0	0
1	1	3
2	1.4	4
3	1.7	3
4	2	0

✓ on calculator



$$\int_0^4 (4t-t^2)\tfrac{1}{2}t^{-1/2}\,dt$$

$$\int_0^4 2t^{1/2} - \tfrac{1}{2}t^{3/2}\,dt$$

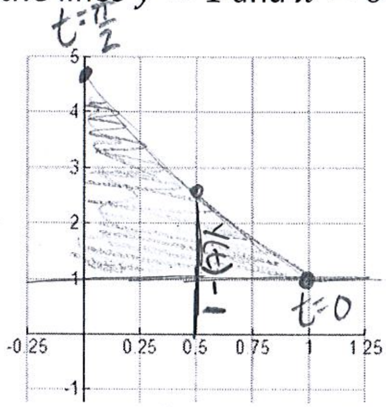
$$\tfrac{2}{3}2t^{3/2} - \tfrac{1}{2}\cdot\tfrac{2}{5}t^{5/2}\Big|_0^4$$

$$\tfrac{4}{3}t^{3/2} - \tfrac{1}{5}t^{5/2} = \tfrac{4}{3}(\sqrt{4})^3 - \tfrac{1}{5}(\sqrt{4})^5 - 0 + 0$$

$$\tfrac{32}{3} - \tfrac{32}{5} = \boxed{\tfrac{64}{15}}$$

Example Four: Consider the curve $C(t) = (cost, e^t)$ on the interval $0 \le t \le \frac{\pi}{2}$. Find the area of the region bounded by $C(t)$ and the lines $y = 1$ and $x = 0$.

t	$\cos t$	e^t
0	1	1
.25	.96	1.2
.5	.87	1.6
.75	.73	2.1
1	.5	2.7
1.25	.3	3.4
$\pi/2 \approx 1.57$	0	4.8



$$\int_{\frac{\pi}{2}}^{0} (e^t - 1)(-\sin t)\,dt$$

$$\frac{e^{\frac{\pi}{2}} - 1}{2} \approx 1.90524$$

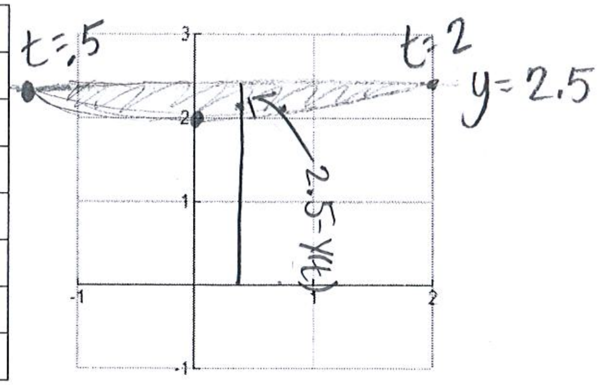
$\boxed{2nd}\,\boxed{7}$

$S((e\wedge t-1)*(-\sin(t)),t,\pi\div 2,0)$ $A = 1.905$

$t - t^{-1}$

Example Five: Find the area of the region enclosed by $y = 2.5$, $x(t) = t - \frac{1}{t}$, & $\boxed{y(t) = t + \frac{1}{t}}$

t	$t - 1/t$	$t + 1/t$
.5	-1.5	2.5
1	0	2
1.5	.83	2.16
2	1.5	2.5



$$2.5 = t + \tfrac{1}{t}$$

$$2.5t = t^2 + 1$$

$$t^2 - 2.5t + 1 = 0$$

$$(t-2)(t-\tfrac{1}{2})$$

$$t = 2 \quad t = \tfrac{1}{2}$$

$$\int_{1/2}^{2} (2.5 - [t + \tfrac{1}{t}])(1 + t^{-2})\,dt$$

use calculator ☺

$.977411$ OR $\dfrac{15 - 16\ln 2}{4}$

OR Solve on calculator

$\boxed{TI-89}$ F2 1: Solve

Solve(2.5=t+1÷t, t)

$\boxed{TI-84}$

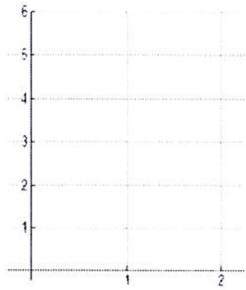
$y_1 = t + \frac{1}{t} - 2.5$

Find x-intercepts

Classwork: Find the area of the region bounded by the parametric curve and the x-axis.

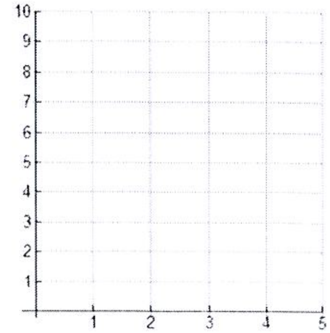
1. $x = 2sin^2\theta$ $\quad$ $y = 2sin^2\theta tan\theta$

$0 \le \theta \le \frac{\pi}{2}$ $\quad$ $Answer$: $\frac{3\pi}{2}$



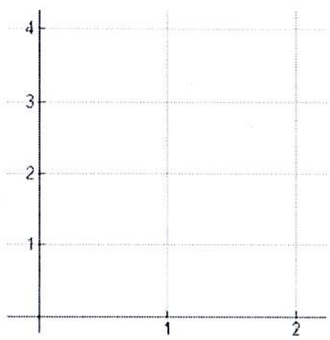
2. $x = t$ $\quad$ $y = 2t$

$0 \le t \le 4$ $\quad$ $Answer$: 16



3. $x = t$ $\quad$ $y = 4 - 2t$

$0 \le t \le 2$ $\quad$ $Answer$: 4



4. $x = t^3$ $\quad$ $y = t + 2$

$1 \le t \le 2$ $\quad$ $Answer$: $\frac{101}{4}$