Slope of the Tangent Line: Let $C(t) = (x(t), y(t))$ where $x(t)$ and $y(t)$ are differentiable. Assume that $x'(t)$ is continuous and $x'(t) \neq 0$. Then

$$\boxed{\frac{dy}{dx}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)} = \frac{\Delta y}{\Delta x} = \frac{\text{Rise}}{\text{Run}}$$

$y = y(t)$
$x = x(t)$

$$= \frac{\frac{dy}{\cancel{dt}}\frac{\cancel{dt}}{dx}}{\frac{\cancel{dx}}{\cancel{dt}}\frac{\cancel{dt}}{\cancel{dx}}}$$

Example One: Let $c(t) = (t^2 + 1, t^3 - 4t)$. Find the equation of the tangent line at $t = 3$ and find the points where the tangent is horizontal.

Tangent at $t=3$

1. point $C(3) = (3^2+1, 3^3-4(3)) = (10, 15)$
2. $m = \frac{y'(t)}{x'(t)} = \frac{3t^2-4}{2t}$

$m(3) = \frac{3(3)^2-4}{2(3)} = \frac{23}{6}$

$$\boxed{y - 15 = \frac{23}{6}(x - 10)}$$

Tangent line

Vertical tangent: set bottom of derivative $=0$ & solve. OR $x'(t)=0$ & solve

Horizontal Tangent: set top of derivative $=0$ & solve OR $y'(t)=0$

$3t^2-4=0$
$3t^2=4$
$t^2=\frac{4}{3}$
$t=\pm\frac{2}{\sqrt{3}}$

$C\left(\frac{2}{\sqrt{3}}\right) = \left(\frac{4}{3}+\frac{3}{3}, \frac{8}{3\sqrt{3}} - 4\left(\frac{2\cdot 3}{3\sqrt{3}}\right)\right)$

$C\left(\frac{2}{\sqrt{3}}\right) = \left(\frac{7}{3}, \frac{-16}{3\sqrt{3}}\right)$

$C\left(-\frac{2}{\sqrt{3}}\right) = \left(\frac{7}{3}, \frac{-8}{3\sqrt{3}} + 4\left(\frac{2\cdot 3}{3\sqrt{3}}\right)\right)$

$C\left(-\frac{2}{\sqrt{3}}\right) = \left(\frac{7}{3}, \frac{16}{3\sqrt{3}}\right)$

The Second Derivative: of a parametric equation $\frac{d^2y}{dx^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)]^3}$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$

Example Two: Find $\frac{d^2y}{dx^2}$, given $x = 8t + 9$, $y = 1 - 4t$, and $t = -3$

$\boxed{x'(t) = 8}$ $y'(t) = -4$
$x''(t) = 0$ $y''(t) = 0$

$$\frac{d^2y}{dx^2} = \frac{8(0) - -4(0)}{8^3} = \frac{0}{\#} = 0$$

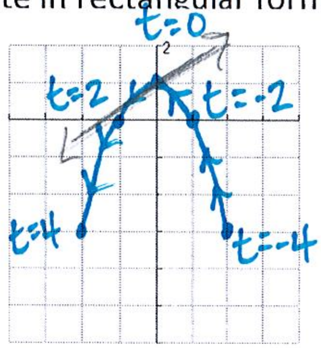
Remember:

Derivative	$f(x)$ The original function
$f'(x) > 0$ pos	f(x) is increasing
$f'(x) < 0$ neg	f(x) is decreasing
$f''(x) > 0$	f(x) is concave up
$f''(x) < 0$	f(x) is concave down

Try It: Given $C(t) = (-\frac{t}{2}, 1 - \frac{t^2}{4})$

A. Graph the function and write in rectangular form.

t	$x = \frac{-t}{2}$	$y = 1 - \frac{t^2}{4}$
-4	2	-3
-2	1	0
0	0	1
2	-1	0
4	-2	-3



Tangent line

$y - \frac{3}{4} = 1(x + \frac{1}{2})$

B. Find the tangent line at $t = 1$

$\boxed{1.}$ Point $C(1) = (\frac{-1}{2}, 1 - \frac{1^2}{4}) = (-\frac{1}{2}, \frac{3}{4})$

$\boxed{2.}$ $m = \dfrac{\frac{-2t}{4}}{\frac{-1}{2}}$ $\quad m(1) = \dfrac{\frac{-1}{2}}{\frac{-1}{2}} = 1$

C. Find the second derivative at $t = -2$

$$\frac{d^2y}{dx^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)]^3}$$

$$= \frac{-\frac{1}{2}(-\frac{1}{2}) - (\frac{-1}{2}t)(0)}{(-\frac{1}{2})^3}$$

$$= \frac{\frac{1}{4}(\frac{8}{-1})}{\frac{-1}{8}\cancel{(\frac{8}{-1})}} = -2$$

$x(t) = -\frac{1}{2}t$ $\quad y(t) = 1 - \frac{1}{4}t^2$

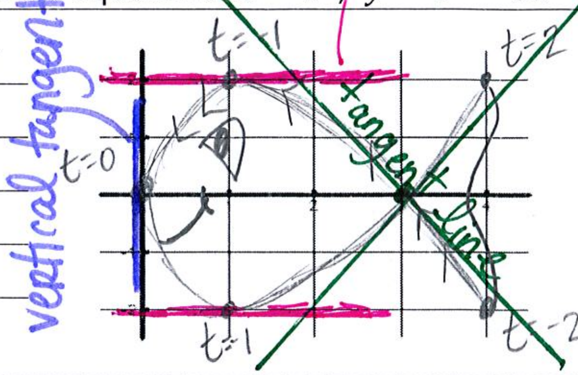
$\boxed{x'(t) = -\frac{1}{2}}$ $\quad y'(t) = -\frac{1}{2}t$

$x''(t) = 0$ $\quad y''(t) = -\frac{1}{2}$

Example Three: A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$

A.) Graph C

t	$x = t^2$	$y = t^3 - 3t$
-2	4	-2
-1	1	2
0	0	0
1	1	-2
2	4	2



B.) Show that C has two tangents at the point (3,0) and find their equations.

Tangent line

$\boxed{1.}$ Point (3,0)

$t=\sqrt{3}$ & $t=-\sqrt{3}$

$\sqrt{x}=\sqrt{t^2}=\sqrt{3}$

$t=\pm\sqrt{3}$

$y=t^3-3t=0$

$t(t^2-3)=0$

$t=0$ $t=\pm\sqrt{3}$

$\boxed{2.}$ Slope $m=\dfrac{3t^2-3}{2t}$

$m(\sqrt{3})=\dfrac{6}{2\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}}=\sqrt{3}$

$m(-\sqrt{3})=\dfrac{6}{2(-\sqrt{3})}\cdot\dfrac{\sqrt{3}}{\sqrt{3}}=-\sqrt{3}$

Tangent line $t=\sqrt{3}$

$y=\sqrt{3}(x-3)$

Tangent line $t=-\sqrt{3}$

$y=-\sqrt{3}(x-3)$

C.) Determine where the curve is concave upward or concave downward.

$x(t)=t^2$ $\quad y(t)=t^3-3t$

$x'(t)=2t$ $\quad y'(t)=3t^2-3$

$x''(t)=2$ $\quad y''(t)=6t$

$\dfrac{d^2y}{dx^2}=\dfrac{2t(6t)-(3t^2-3)(2)}{[2t]^3}$

$\dfrac{d^2y}{dx^2}=\dfrac{12t^2-6t^2+6}{8t^3}$

$\dfrac{d^2y}{dx^2}=\dfrac{6t^2+6}{8t^3}$ ← always positive

$8t^3=0$ $t=0$

$-\;|\;+$ (t=0)

concave down when $t<0$

concave up when $t>0$

D.) Determine if the graph has any vertical tangents or horizontal tangents and list the points where they are located.

Horizontal Tangents

$y'(t)=0$ & solve

$3t^2-3=0$

$3t^2=3$

$\sqrt{t^2}=\sqrt{1}$

$t=1$ & $t=-1$

$(1,-2)$ & $(1,2)$

Vertical Tangents

$x'(t)=0$

$2t=0$

$t=0$

$(0,0)$

Proof of

$$\frac{d^2y}{dx^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)]^3}$$

$x = t + \cos t$

$y = \sin t$

$$\frac{d}{dt}[y = \sin t]$$

$$\frac{d}{dt}[x = t + \cos t]$$

$$\frac{\cancel{\frac{dt}{dx}}\frac{dy}{\cancel{dt}} = \cos t}{\cancel{\frac{dt}{dx}}\boxed{\frac{dx}{\cancel{dt}} = 1 - \sin t}}$$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \left[\frac{\cos t}{1 - \sin t}\right]\frac{d}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{(1-\sin t)\frac{d}{dx}(\cos t) - \cos t\frac{d}{dx}(1-\sin t)}{(1-\sin t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(1-\sin t)(-\sin t)\frac{dt}{dx} - \cos t(-\cos t)\frac{dt}{dx}}{(1-\sin t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\cancel{\frac{dx}{dt}}\cancel{\frac{dt}{dx}}\left[(1-\sin t)(-\sin t) - \cos t(-\cos t)\right]}{\boxed{\frac{dx}{dt}}(1-\sin t)^2}$$

$$\frac{d^2y}{dx^2} \quad \frac{-\sin t + \sin^2 t + \cos^2 t}{(1-\sin t)(1-\sin t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\sin t + 1}{(1-\sin t)^3}$$

$x = x(t)$

$y = y(t)$

$$\frac{d}{dt}[y = y(t)]$$

$$\frac{d}{dt}[x = x(t)]$$

$$\frac{\frac{dy}{\cancel{dt}}\cancel{\frac{dt}{dx}} = y'(t)}{\frac{dx}{\cancel{dt}}\cancel{\frac{dt}{dx}} = x'(t)}$$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \left[\frac{y'(t)}{x'(t)}\right]\frac{d}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{x'(t)\frac{d}{dx}[y'(t)] - y'(t)\frac{d}{dx}[x'(t)]}{[x'(t)]^2}$$

$$\frac{d^2y}{dx^2} = \frac{x'(t)y''(t)\cdot\frac{dt}{dx} - y'(t)x''(t)\cdot\frac{dt}{dx}}{[x'(t)]^2}$$

$$\frac{d^2y}{dx^2} \quad \frac{\cancel{\frac{dx}{dt}}\cancel{\frac{dt}{dx}}[x'(t)y''(t) - y'(t)x''(t)]}{\frac{dx}{dt}[x'(t)]^2}$$

$$\frac{d^2y}{dx^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)][x'(t)]^2}$$

$$\frac{d^2y}{dx^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)]^3}$$

Simplify :)

$$\frac{d^2y}{dx^2} = \frac{1}{(1-\sin t)^2}$$

we got the same answer in class just using (this) formula :)