Notes: Taylor Polynomials & Error Bound Day 7

Taylor Polynomial: centered at $x = c$

$$T_n(x) = \frac{f(c)}{0!} + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots \ldots \ldots \frac{f^n(c)}{n!}(x-c)^n$$

centered at a

Example One: Compute $T_2(x)$ for $f(x)$ centered at $x = a$ and use a calculator to compute the error $|f(x) - T_2(x)|$ at the given value of x.

Actual - Approx

a. $f(x) = e^x$, $x = .25$, about $a = 0$ ERROR $= .003$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $T_2(x) = 1 + x + \frac{1}{2}x^2$

$e^{.25} \approx T_2(.25) = 1 + \frac{1}{4} + \frac{1}{2}\left(\frac{1}{4}\right)^2 = \frac{32}{32} + \frac{8}{32} + \frac{1}{32} = \frac{41}{32} =$ Approx ≈ 1.281

Actual $f(.25) = e^{.25} = 1.284$ ERROR $= |f(.25) - T_2(.25)| = |1.284 - 1.281|$

$f(x) = sinx$, $x = \frac{\pi}{15}$, about $a = 0$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $T_2(x) = x$ Approx $T_2\left(\frac{\pi}{15}\right) = .20944$

$f\left(\frac{\pi}{15}\right) = \sin\left(\frac{\pi}{15}\right) \approx .207912$

(Activity) ERROR $= |.20944 - .207912| = .0015278$

Lagrange Error Bound:

Suppose $f^{(n+1)}(x)$ exists and is continuous for all values between x and a. Let K be a value such that $|f^{(n+1)}(u)| \le K$ for all u between x and a.

Then $\underbrace{|f(x) - T_n(x)|}_{\text{ERROR}} \le \underbrace{\frac{K}{(n+1)!}|x-a|^{n+1}}_{\text{Max ERROR}}$

n = degree
a = centered
x = what you are approximating

To find K: $\boxed{1.}$ You find the n+1 derivative
$\boxed{2.}$ $|f^{n+1}(a)|$ & $|f^{n+1}(x)|$
$\boxed{3.}$ The larger of step 2 = K

Example Three: Use the error bound to find a bound for the error $|T_3(1.2) - \ln(1.2)|$, where $T_3(x)$ is the 3rd Taylor Polynomial for $f(x) = lnx$ at $a = 1$.

$T_3(x) = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$

$T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

$T_3(1.2) = .2 - \frac{1}{2}(.2)^2 + \frac{1}{3}(.2)^3 = .182667$

ERROR

$|T_3(1.2) - \ln(1.2)|$

$|.182667 - \ln(1.2)|$

$.000345 \le .0004$

Max ERROR

$\frac{K}{(n+1)!}|x-a|^{n+1}$

$\frac{K}{4!}|1.2-1|^4$

$\frac{6}{24}(.2)^4$

$.0004$

Find K

$\boxed{1.}$ $f(x) = \ln x$ $f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $f'(1) = 1$

$f''(x) = -1x^{-2}$ $f''(1) = -1$

$f^3(x) = 2x^{-3}$ $f^3(1) = 2$

$f^4(x) = -6x^{-4}$

$\boxed{2}$ $|f^4(1)|$ & $|f^4(1.2)|$

$\left|\frac{-6}{(1)^4}\right| = 6$ $\left|\frac{-6}{(1.2)^4}\right| = 2.8$

$\boxed{3.}$ $K = 6$

Day 7 Activity

Example One: Given $f(x)=e^x$

A.) State the general Maclaurin for $f(x)=e^x$ state the first 5 terms.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

B.) On the same graph, graph each of the following

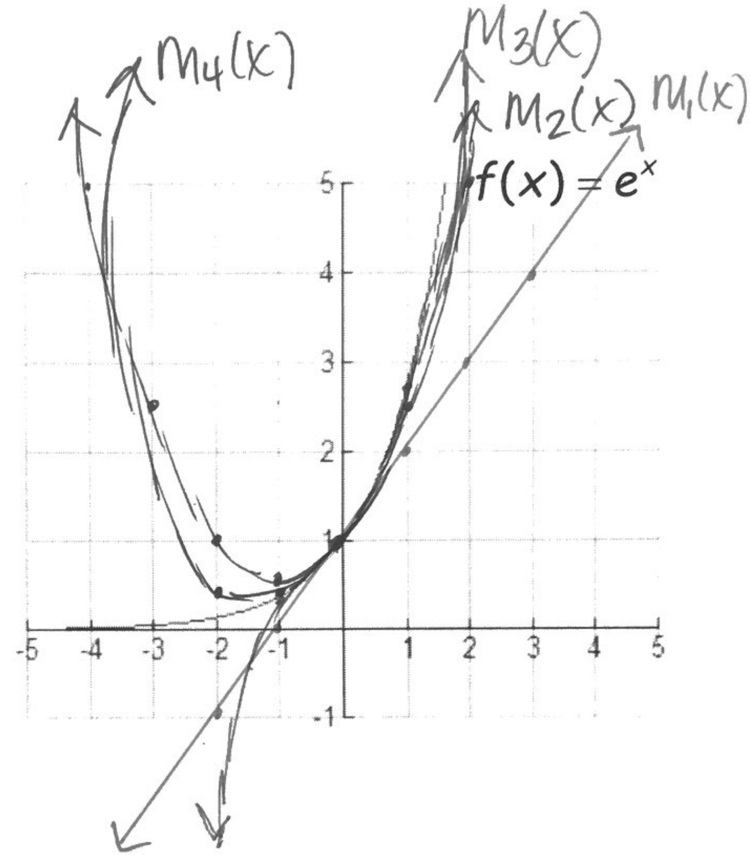
$f(x)=e^x$

$M_1(x) = 1+x$
linear

$M_2(x) = 1+x+\frac{1}{2}x^2$
quadratic

$M_3(x) = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$
cubic

$M_4(x) = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4$
quartic



C.) Use $M_4(x)$ to approximate each:

Actual	$e^x \approx M_4(x) \approx$ Approximate
$e^1 \approx 2.718$	$e^1 \approx 1+(1)+\frac{1}{2}(1)^2+\frac{1}{6}(1)^3+\frac{1}{24}(1)^4 = 2.708$
$e^2 \approx 7.389$	$e^2 \approx 1+(2)+\frac{1}{2}(2)^2+\frac{1}{6}(2)^3+\frac{1}{24}(2)^4 = 7$
$e^3 \approx 20.085$	$e^3 \approx 1+(3)+\frac{1}{2}(3)^2+\frac{1}{6}(3)^3+\frac{1}{24}(3)^4 = 16.375$
$e^4 \approx 54.5982$	$e^4 \approx 1+(4)+\frac{1}{2}(4)^2+\frac{1}{6}(4)^3+\frac{1}{24}(4)^4 = 34.\bar{3}$

D.) What is the error for $|e^2 - M_3(2)|$ $\quad |e^x - M_n(x)|$

$$\left|e^2 - \left(1+(2)+\tfrac{1}{2}(2)^2+\tfrac{1}{6}(2)^3\right)\right|$$

1.0557

E.) What is the Maximum error for $|e^2 - M_3(2)|$

$n=3 \quad x=2 \quad$ centered $a=0$

$$\frac{K}{(n+1)!}|x-a|^{n+1}$$

$$\frac{K}{4!}|2-0|^4$$

$$\frac{e^2}{24}(2)^4 = \frac{2}{3}e^2 = 4.926$$

$\boxed{1.}$ $f(x)=e^x$
$f'(x)=e^x$
$f''(x)=e^x$
$f^3(x)=e^x$
$f^4(x)=e^x$

$\boxed{2}$ $|f^4(0)|$ & $|f^4(2)|$
$=1 \qquad = e^2$

$\boxed{3}$ $K=e^2$