AP Calculus BC — Name________________

Taylor & Radius of Convergence — Power Day 5

1. Write the first four terms of the Taylor series of f(x) centered at c = 3 if

$f(3) = 1, \quad f'(3) = 2, \quad f''(3) = 12, \quad f'''(3) = 3$

$$= 1 + 2(x-3) + \frac{12(x-3)^2}{2} + \frac{3(x-3)^3}{3!}$$

$$= 1 + 2(x-3) + 6(x-3)^2 + \frac{(x-3)^3}{2}$$

2. Find the Taylor series for f centered at 4 if $f^{n}(4) = \frac{(-1)^n 2^n}{n}$. What is the radius of convergence?

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n (x-4)^n}{n \cdot n!}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} 2^{n+1} (x-4)^{n+1}}{(n+1)(n+1)!}}{\frac{(-1)^n 2^n (x-4)^n}{n \cdot n!}} \right| = \lim_{n\to\infty} \left| \frac{2^n \cdot 2 (x-4)^n (x-4)}{(n+1)(n+1)n!} \cdot \frac{n\, n!}{2^n (x-4)^n} \right|$$

$$= \lim_{n\to\infty} \frac{2n}{n^2+2n+1} |x-4| = 0|x-4| < 1$$

$$(-\infty, \infty)$$

3-8: Find the Taylor series for f(x) centered at the given value of c. [Assume that f has a power series expansion.] Also find the associated radius of convergence.

3. $f(x) = \ln x$, $c = 2$

$f(x) = \ln x \qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$

$f''(x) = -\frac{1}{x^2} \qquad f''(2) = -\frac{1}{4}$

$f'''(x) = \frac{2}{x^3} \qquad f'''(2) = \frac{2}{8} = \frac{1}{4}$

$f^{(4)}(x) = \frac{-6}{x^4} \qquad f^{(4)}(2) = \frac{-6}{16} = \frac{-3}{8}$

$$f(x) = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{4}\frac{(x-2)^2}{2!} + \frac{1}{4}\frac{(x-2)^3}{3!} - \frac{3}{8}\frac{(x-2)^4}{4!} + \ldots$$

$$= \ln 2 + \frac{(x-2)}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} - \frac{(x-2)^4}{64}$$

$1 \quad 2\cdot 4 \quad 3\cdot 8 \quad 4\cdot 16$

$$= \boxed{\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n \cdot 2^n}}$$

$R = 2$

$\boxed{0 < x \le 4}$

$x = 0 \quad \sum \frac{(-1)^n (-2)^n}{n 2^n} = \sum \frac{1}{n}$ D p-series

$x = 4 \quad \sum \frac{(-1)^n 2^n}{n 2^n}$ C alt series

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-2)^{n+1}}{(n+1)2^n \cdot 2} \cdot \frac{n \cdot 2^n}{(-1)^n (x-2)^n} \right| = \lim_{n\to\infty} \frac{n}{2n+2}|x-2| = \frac{1}{2}|x-2| < 1$$

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4. $f(x) = \frac{1}{x}$, $c = -3$

$f(x) = x^{-1} \qquad f(-3) = -\frac{1}{3}$

$f'(x) = -x^{-2} \qquad f'(-3) = \frac{-1}{9}$

$f''(x) = 2x^{-3} \qquad f''(-3) = \frac{-2}{27}$

$f'''(x) = -6x^{-4} \qquad f'''(-3) = \frac{-6}{81}$

$f^{(n)}(x) = 24x^{-5} \qquad f^{(4)}(-3) = \frac{-24}{243}$

$$f(x) = -\frac{1}{3} - \frac{1}{9}(x+3) - \frac{2}{27}\frac{(x+3)^2}{2} - \frac{6}{81}\frac{(x+3)^3}{3!} - \frac{24}{243}\frac{(x+3)^4}{4!} - \ldots$$

$$= -\frac{1}{3} - \frac{(x+3)}{9} - \frac{(x+3)^2}{27} - \frac{(x+3)^3}{81} - \frac{(x+3)^4}{243} - \ldots$$

$$= \sum_{n=0}^{\infty} -\frac{(x+3)^n}{3^{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{\frac{-(x+3)^{n+1}}{3^{n+2}}}{\frac{-(x+3)^n}{3^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(x+3)^n (x+3)}{3^n \cdot 3^2} \cdot \frac{3^n \cdot 3}{(x+3)^n} \right|$$

$$= \lim_{n\to\infty} \frac{3}{3^2}|x+3| = \frac{1}{3}|x+3| < 1$$

$\underline{R = 3}$

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5. $f(x)=e^{2x}$, $c=3$

$f(x)=e^{2x}$ $f(3)=e^6$

$f'(x)=2e^{2x}$ $f'(3)=2e^6$

$f''(x)=4e^{2x}$ $f''(3)=4e^6$

$f'''(x)=8e^{2x}$ $f'''(3)=8e^6$

$f^{(4)}(x)=16e^{2x}$ $f^{(4)}(3)=16e^6$

$$f(x)=e^6+2e^6(x-3)+\frac{4e^6(x-3)^2}{2}+\frac{8e^6(x-3)^3}{3!}+\frac{16e^6(x-3)^4}{4!}+\dots$$

$$=\sum_{n=0}^{\infty}\frac{2^n e^6(x-3)^n}{n!}$$

$$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}e^6(x-3)^{n+1}}{(n+1)!}}{\frac{2^n e^6(x-3)^n}{n!}}\right|=\lim_{n\to\infty}\left|\frac{\cancel{2^n}\cdot 2\cdot \cancel{e^6}\cancel{(x-3)^n}(x-3)}{(n+1)\cancel{n!}}\cdot\frac{\cancel{n!}}{\cancel{2^n}\cancel{e^6}\cancel{(x-3)^n}}\right|$$

$$=\lim_{n\to\infty}\frac{2}{n+1}|x-3|=0|x-3| \quad (-\infty,\infty) \quad R=\infty$$

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6. $f(x)=\sin x$, $c=\frac{\pi}{2}$

$f(x)=\sin x$ $f(\pi/2)=1$

$f'(x)=\cos x$ $f'(\pi/2)=0$

$f''(x)=-\sin x$ $f''(\pi/2)=-1$

$f'''(x)=-\cos x$ $f'''(\pi/2)=0$

$f^{(4)}(x)=\sin x$ $f^{(4)}(\pi/2)=1$

$$f(x)=1+0(x-\pi/2)-\frac{1(x-\pi/2)^2}{2!}+\frac{0(x-\pi/2)^3}{3!}+\frac{1(x-\pi/2)^4}{4!}-\dots$$

$$=1-\frac{(x-\pi/2)^2}{2!}+\frac{(x-\pi/2)^4}{4!}-\frac{(x-\pi/2)^6}{6!}+\dots \qquad R=\infty$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi/2)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(x-\frac{\pi}{2})^{2(n+1)}}{(2(n+1))!}}{\frac{(-1)^n(x-\pi/2)^{2n}}{(2n)!}}\right|=\lim_{n\to\infty}\left|\frac{\cancel{(x-\frac{\pi}{2})^{2n}}(x-\frac{\pi}{2})^2\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\cancel{(x-\frac{\pi}{2})^{2n}}}\right|$$

$$=\lim_{n\to\infty}\frac{1}{4n^2+6n+2}\left|x-\frac{\pi}{2}\right|^2$$

$=0<1$

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7. $f(x)=\cos x$, $c=\pi$

$f(x)=\cos x$ $f(\pi)=-1$

$f'(x)=-\sin x$ $f'(\pi)=0$

$f''(x)=-\cos x$ $f''(\pi)=1$

$f'''(x)=\sin x$ $f'''(\pi)=0$

$f^{(4)}(x)=\cos x$ $f^{(4)}(\pi)=-1$

$$f(x)=-1+0(x-\pi)+\frac{1(x-\pi)^2}{2}+\frac{0(x-\pi)^3}{3!}-\frac{1(x-\pi)^4}{4!}+\dots$$

$$=-1+\frac{(x-\pi)^2}{2}-\frac{(x-\pi)^4}{4!}+\frac{(x-\pi)^6}{6!}-\dots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-\pi)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(x-\pi)^{2n+2}}{(2n+2)!}}{\frac{(-1)^{n+1}(x-\pi)^{2n}}{(2n)!}}\right|=\lim_{n\to\infty}\left|\frac{\cancel{(x-\pi)^{2n}}(x-\pi)^2}{(2n+2)(2n+1)\cancel{(2n)!}}\cdot\frac{\cancel{(2n)!}}{\cancel{(x-\pi)^{2n}}}\right|$$

$$=\lim_{n\to\infty}\frac{1}{4n^2+6n+2}|x-\pi|^2=0<1 \qquad R=\infty$$

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8. $f(x)=\frac{1}{1-x}$, $c=2$

$f(x)=(1-x)^{-1}$ $f(2)=-1$

$f'(x)=(1-x)^{-2}$ $f'(2)=1$

$f''(x)=2(1-x)^{-3}$ $f''(2)=-2$

$f'''(x)=6(1-x)^{-4}$ $f'''(2)=6$

$f^{(4)}(x)=24(1-x)^{-5}$ $f^{(4)}(2)=-24$

$$f(x)=-1+1(x-2)-\frac{2(x-2)^2}{2}+\frac{6(x-2)^3}{3!}-\frac{24(x-2)^4}{4!}+\dots$$

$$=-1+(x-2)-(x-2)^2+(x-2)^3-(x-2)^4+\dots$$

$$=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{(-1)^{n+1}(x-2)^n}\right|=\lim_{n\to\infty}|x-2|<1 \qquad R=1$$