Interval of Convergence

General Form of a Power Series:

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 \dots a_n(x-c)^n = \sum_{n=0}^{\infty} a_n(x-c)^n$$

Convergence of a Power Series:

Power series are only valid on the intervals for which they converge. We define the following two terms to discuss convergence of a power series. $(0,2)$ $[-1,9]$ $(-\sqrt{2},\sqrt{2}]$

- Interval of convergence (I)- range of x —values for which a power series will converge
- Radius of convergence (R)- distance from the center, $x = c$, to the edge of the interval of convergence

$R=1$ $R=5$ $R=\sqrt{2}$

Radius of Convergence: Let $f(x) = a(x-c)^n$. The 3 possible results involving convergence of power series are

Ratio Test | Answer | I.O.C

1. Converges only when x = c — $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty \to R = 0 \to I = \{c\}$
2. Converges for all x — $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1 \to R = \infty \to I = (-\infty,\infty)$
3. There is a positive number R such that the series converges absolutely if |x – c| < R — $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \underset{\text{some number}}{a|x-c|^n} = \to R = \epsilon\mathbb{R}^+ \to I$ (must check endpoints)

Solve like yesterday

To find the interval of convergence, we can use the Ratio Test

- Leaving the x in the limit you get the interval of convergence
- Pulling the x out you get $\frac{1}{R}$ where R=the radius of convergence.

$-2 < x < 2$

I.O.C $(-2,2)$ R.O.C 2

Example One: Using the Ratio Test: For which values of x does this function converge?

$$\sum_{n=0}^{\infty}\frac{x^n}{2^n} \quad R = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^n}\cdot x^1\cdot\cancel{2^n}}{\cancel{x^n}\cdot\cancel{2^n}\cdot 2^1}\right| = \lim_{n\to\infty}\left|\frac{x}{2}\right|$$

$$R = |x|\lim_{n\to\infty}\left|\frac{1}{2}\right|$$

$$R = |x|\left(\frac{1}{2}\right) < 1$$ And solve for x

$|x| < 2$

$x < 2$ and $x > -2$

$\boxed{\text{Check }-2}$ $\sum_{n=0}^{\infty}\frac{x^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty}\left(\frac{-2}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n$

diverges by geometric $|R| = 1 \geq 1$.

$\boxed{\text{check } 2}$ $\sum_{n=0}^{\infty}\frac{(2)^n}{(2)^n} = \sum_{n=0}^{\infty}(1)^n$

diverges by geometric $|R| = 1 \geq 1$

$4 < x \leq 6$

I.O.C. = $(4,6]$ R.O.C. = 1

Example Two: Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-5)^n$$

$$R = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)}\cdot\frac{n}{(x-5)^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(x-5)^n}(x-5)^1\, n}{\cancel{(x-5)^n}(n+1)}\right|$$

$$R = |x-5|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$R = |x-5|(1) < 1$$

$\boxed{\text{solve for } x}$

$x-5<1$ And $x-5>-1$

$x<6$ and $x>4$

$\boxed{\text{Check } 4}$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(4-5)^n = \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$

diverges by p-series $p=1\leq 1$.

$\boxed{\text{check } 6}$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(6-5)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n(1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

converges by alternating b.c. $\frac{1}{n}$ is decreasing & $\displaystyle\lim_{n\to\infty}\frac{1}{n}=0$

Example Three: Determine the convergence of

$$\sum_{n=1}^{\infty}\frac{x^n}{n!}$$

$$R = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^n}\cdot x^1\cdot \cancel{n!}}{\cancel{x^n}(n+1)\cdot\cancel{n!}}\right|$$

$$R = |x|\lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$R = |x|(0) < 1 \leftarrow$ Always true

I.O.C $= (-\infty,\infty)$ R.O.C $= \infty$

Example Four: Determine the convergence of

$$\sum_{k=1}^{\infty} \frac{(x-3)^k}{k4^{k+1}}$$

$$R = \lim_{k\to\infty}\left|\frac{(x-3)^{k+1}}{(k+1)\cdot 4^{k+2}}\cdot\frac{k4^{k+1}}{(x-3)^k}\right| = \lim_{k\to\infty}\left|\frac{\cancel{(x-3)^k}(x-3)^1\cancel{4^k}\cancel{4^1}(k)}{\cancel{(x-3)^k}\cancel{4^k}4^2(k+1)}\right|$$

$$R = |x-3|\lim_{k\to\infty}\left|\frac{k}{4(k+1)}\right|$$

$$R = |x-3|\left(\frac{1}{4}\right) < 1$$

$|x-3| < 4$

$x-3 < 4$ and $x-3 > -4$

$x < 7$ and $x > -1$

R.O.C = $\underline{4}$ I.O.C = $\underline{[-1,7)}$

$\boxed{\text{Check } -1}$ $\sum_{k=1}^{\infty}\frac{(-4)^k}{k\cdot 4^k\cdot 4^1} = \sum_{k=1}^{\infty}\left(\frac{-4}{4}\right)^k\cdot\frac{1}{4k} = \sum_{k=1}^{\infty}\frac{(-1)^k}{4k}$ Converges by Alternating $\frac{1}{4k}$ is decreasing & $\lim_{k\to\infty}\frac{1}{4k} = 0$

$\boxed{\text{check } 7}$ $\sum_{k=1}^{\infty}\frac{\cancel{4^k}}{k\cdot\cancel{4^k}\cdot 4^1} = \sum_{k=1}^{\infty}\frac{1}{4k} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k}$ diverges by p-series $p=1\leq 1$.

Example Five: Determine the convergence of

$$\sum_{n=1}^{\infty} n!\,(2x+1)^n$$

$$R = \lim_{n\to\infty}\left|\frac{(n+1)!\,(2x+1)^{n+1}}{n!\,(2x+1)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cancel{n!}\cancel{(2x+1)^n}(2x+1)^1}{\cancel{n!}\cancel{(2x+1)^n}}\right|$$

$$R = \lim_{n\to\infty}|(n+1)(2x+1)| = |(2x+1)|\lim_{n\to\infty}|n+1|$$

$$R = |2x+1|\,\infty < 1$$

$2x+1=0$

$2x=-1$

$x=-\frac{1}{2}$

R.O.C = $\underline{0}$

I.O.C = $\underline{\{-\tfrac{1}{2}\}}$

Example Six: Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}$$

$$R = \lim_{n\to\infty} \left| \frac{x^{(2)(n+1)}}{4^{n+1}\,[(n+1)!]^2} \cdot \frac{4^n\,[(n)!]^2}{x^{2n}} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{\cancel{x^{2n}} \cdot x^2 \cdot \cancel{4^n}\, \cancel{n!} \cdot \cancel{n!}}{\cancel{x^{2n}} \cdot \cancel{4^n} \cdot 4^1 (n+1) \cancel{n!} (n+1) \cancel{n!}} \right|$$

$$R = |x^2| \lim_{n\to\infty} \left| \frac{1}{4(n+1)^2} \right|$$

$|x^2|(0) < 1$ always ☺

I.O.C $= (-\infty, \infty)$ R.O.C $= \infty$