Review #1 Pg. 514 20, 22, 25, 26, 27, 28, 31, 33
620 91, 93A, 96, 97, 98

Find the Taylor Polynomial at $x=a$ for the given function.

20. $f(x)=x^3$ $T_3(x)$, $a=1$

$f(x)=x^3$ $f(1)=1^3=1$
$f'(x)=3x^2$ $f'(1)=3(1)^2=3$
$f''(x)=6x$ $f''(1)=6(1)=6$
$f'''(x)=6$ $f'''(1)=6$

$T_3(x)=\frac{1}{0!}(x-1)^0+\frac{3}{1!}(x-1)^1+\frac{6}{2!}(x-1)^2+\frac{6}{3!}(x-1)^3$

$T_3(x)=1+3(x-1)+3(x-1)^2+(x-1)^3$ ✓

22. $f(x)=x\ln x$ $T_4(x)$, $a=1$

$f(x)=x\ln(x)$ $f(1)=1\ln(1)=0$
$f'(x)=x\frac{1}{x}+\ln(x)(1)=1+\ln x$ $f'(1)=1+\ln(1)=1$
$f''(x)=1/x=x^{-1}=1/x$ $f''(1)=1/1=1$
$f'''(x)=-x^{-2}=-1/x^2$ $f'''(1)=-1/1^2=-1$
$f''''(x)=2x^{-3}=2/x^3$ $f^4(1)=2/1^3=2$

$T_4(x)=\frac{0}{0!}(x-1)^0+\frac{1}{1!}(x-1)^1+\frac{1}{2!}(x-1)^2-\frac{1}{3!}(x-1)^3+\frac{2}{4!}(x-1)^4$

$T_4(x)=0+(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{6}(x-1)^3+\frac{1}{12}(x-1)^4$ ✓

25. $f(x)=\ln(\cos x)$ $T_3(x)$ $a=0$

$f(x)=\ln(\cos x)$ $f(0)=\ln(\cos(0))=\ln(1)=0$
$f'(x)=\frac{1}{\cos x}\cdot -\sin x=-\tan x$ $f'(0)=-\tan(0)=-0=0$
$f''(x)=-\sec^2 x$ $f''(0)=-(\sec(0))^2=-(1)^2=-1$
$f'''(x)=-2\sec x\cdot\sec x\tan x=-2\sec^2 x\tan x$ $f'''(1)=-2(\sec 0)^2\tan 0$
$=-2(1)^2(0)=0$

$T_3(x)=-\frac{1}{2!}(x-0)^2=-\frac{1}{2}x^2$ ✓

26. Find the n^{th} Maclaurin polynomial for $f(x)=e^{3x}$

$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$e^{3x}=\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}=\sum_{n=0}^{\infty}\frac{3^n}{n!}x^n$ ✓

27. Use the 5^{th} Maclaurin Polynomial of $f(x)=e^x$ to approximate $\sqrt{e}$. Use a calculator to determine the ERROR

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^x = \sqrt{e} = e^{1/2} \text{ So } x = 1/2$$

$$M_5 = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$$

$$M_5\left(\tfrac{1}{2}\right) = 1 + \tfrac{1}{2} + \tfrac{1}{2}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{6}\left(\tfrac{1}{2}\right)^3 + \tfrac{1}{24}\left(\tfrac{1}{2}\right)^4 + \tfrac{1}{120}\left(\tfrac{1}{2}\right)^5 = 1.6487$$

$$e^{1/2} = 1.64872$$

$$\text{ERROR} = |1.64872 - 1.6487| = .000021 \checkmark$$

28. Use the third Taylor Polynomial of $f(x) = \tan^{-1}x$ at $a=1$ to approximate $f(1.1)$. Use a calculator to determine the ERROR.

$$f(x) = \tan^{-1}x \qquad f(1) = \tfrac{\pi}{4} \quad f'(1) = 1/2 \quad f''(1) = -2/4 = -1/2 \quad f'''(x) = 4/8 = \tfrac{1}{2}$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = \frac{(1+x^2)(0)-(1)(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

$$f'''(x) = \frac{(1+x^2)^{\cancel{2}}(-2)-(-2x)2\cancel{(1+x^2)}(2x)}{(1+x^2)^{\cancel{4}3}} = \frac{-2(1+x^2)+8x^2}{(1+x^2)^3}$$

$$f'''(x) = \frac{-2-2x^2+8x^2}{(1+x^2)^3} = \frac{6x^2-2}{(1+x^2)^3}$$

$$T_3 = \frac{\pi}{4}\frac{(x-1)^0}{0!} + \frac{1/2}{1!}(x-1)^1 - \frac{\cancel{1/2}\,1/2}{\cancel{2!}}(x-1)^2 + \frac{\cancel{\tfrac{1}{2}}\,\tfrac{1}{6}}{\cancel{3!}}(x-1)^3$$

$$T_3 = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3$$

$$T_3(1.1) = \frac{\pi}{4} + \frac{1}{2}(.1) - \frac{1}{4}(.1)^2 + \frac{1}{12}(.1)^3 = .832981496$$

$$\tan^{-1}(1.1) = .832981266$$

$$\text{ERROR} = |.832981 - .832981| = .0000002301$$

31. Let $T_4(x)$ be the Taylor polynomial for $f(x) = x\ln x$ at $x=1$ computed in Exercise 22. Use the error bound to find a bound for $|f(1.2) - T_4(1.2)|$

$\boxed{\text{Error Bound}}$

$$\frac{|K|}{(n+1)!}|x-a|^{n+1}$$

$$\frac{|K|}{(4+1)!}|1.2-1|^{4+1}$$

$$\frac{6}{5!}|.2|^5$$

$$= \boxed{.000016}$$

is the largest error the $T_4(x)$ could have

To Find K:

$f(x) = x\ln x$

$f'(x) = 1+\ln x$

$f''(x) = 1/x$

$f^3(x) = -1/x^2$

$f^4(x) = 2/x^3$

$f^5(x) = -6/x^4$

$f^5(1) = \frac{-6}{(1)^4}$ $f^5(1.2) = \frac{-6}{(1.2)^4}$

Bigger $\Longrightarrow \boxed{=6}$ $= 2.89$

33. Show that the nth Maclaurin polynomial for $f(x) = \frac{1}{1-x}$ is $T_n(x) = 1 + x + x^2 + \dots + x^2$. Conclude by substituting $x/4$ for x so that the nth Maclaurin polynomial for $f(x) = \frac{1}{1-x/4}$ is $T_n(x) = 1 + \frac{1}{4}x + \frac{1}{4^2}x^2 + \dots + \frac{1}{4^n}x^n$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-\frac{x}{4}} = \sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{1\cdot x^n}{4^n} = \sum_{n=0}^{\infty}\frac{1}{4^n}x^n$$

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91. Expand the function $f(x) = \frac{2}{4-3x}$ as a power series centered at $c=0$. Determine the values of x for which the series converges.

$$f(x) = \frac{2}{4-3x} = 2\left(\frac{1}{4-3x}\right) = 2\left(\frac{1}{4(1-\frac{3}{4}x)}\right) = \frac{2}{4}\left(\frac{1}{1-\frac{3}{4}x}\right) = \frac{1}{2}\left(\frac{1}{1-\frac{3}{4}x}\right)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{2}\cdot\frac{1}{1-\frac{3}{4}x} = \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{3x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^1}\cdot\frac{3^n x^n}{(2^2)^n} = \sum_{n=0}^{\infty}\frac{3^n}{2^{2n+1}}x^n$$

$$r = \lim_{n\to\infty}\left|\frac{3^{n+1}}{2^{2n+2}}\cdot\frac{2^{2n+1}}{3^n}\right| = \frac{3}{2}$$

$$|x| < \frac{2}{3}$$

93A. Let $F(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k \cdot k!} = \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} x^{2k}$

A. Show that $F(x)$ has infinite Radius of convergence

$$r = \lim_{k \to \infty} \left| \frac{\frac{1}{2^{k+1}(k+1)!}}{\frac{1}{2^k k!}} \right| = \lim_{k \to \infty} \frac{1 \cdot \cancel{2^k} \cdot \cancel{k!}}{\cancel{2^k} 2^1 (k+1) \cancel{k!}} = \lim_{k \to \infty} \frac{1}{2(k+1)} = 0$$

when $R=0$ the Radius of convergence is $(-\infty, \infty)$

Find the Taylor series centered at c

96. $f(x) = e^{4x}$, $c=0$

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{4x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = \sum_{n=0}^{\infty} \frac{4^n x^n}{n!}$$

97. $f(x) = e^{2x}$, $c=-1$

$$f(x) = e^{2(x+1)-2}$$

$$= e^{2(x+1)} e^{-2}$$

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{e^{-2}(2(x+1))^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{e^2 n!}(x+1)^n$$

98. $f(x) = x\sin x$, $C = \pi$

$f(\pi) = \pi \sin(\pi) = 0$

$f'(x) = x\cos x + \sin x(1)$

$f''(x) = x(-\sin x) + \cos x(1) + \cos x$

$f''(x) = -x\sin x + 2\cos x$

$f^3(x) = -x\cos x + \sin x(-1) + 2(-\sin x)$

$f^3(x) = -x\cos x - 3\sin x$

$f^4(x) = -x(-\sin x) + \cos x(-1) - 3\cos x$

$f^4(x) = x\sin x - 4\cos x$

$f'(\pi) = \pi(-1) + 0 = -\pi$

$f''(\pi) = -\pi(0) + 2\cos\pi = -2$

$f^3(\pi) = -\pi(-1) - 3(0) = \pi$

$f^4(\pi) = \pi(0) - 4(-1) = 4$

$f^5(\pi) = -\pi$

$f^6(\pi) = -6$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi (x-\pi)^{n+1}}{(n+1)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n)}{(2n)!}$$