AP Calculus BC

Name________________________

Supplement Derivatives of Polar Functions

P, P, & V Day 9

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}=\frac{r'(\theta)\sin\theta+r\cos\theta}{r'(\theta)\cos\theta-r\sin\theta}$$

1. Find the slope of the tangent line to $r=\theta$ at $\theta=\frac{\pi}{2}$ and $\theta=\pi$ $\frac{d}{d\theta}[r=\theta]$ $r'=1$

$$\frac{dy}{dx}=\frac{(1)\sin\theta+\theta\cos\theta}{(1)\cos\theta-\theta\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\frac{\pi}{2}}=\frac{\sin\frac{\pi}{2}+\cancel{\frac{\pi}{2}\cos\frac{\pi}{2}}\to 0}{\cancel{\cos\frac{\pi}{2}}\to 0-\frac{\pi}{2}\sin\frac{\pi}{2}}=\frac{1}{-\frac{\pi}{2}}=\boxed{-\frac{2}{\pi}}$$

$$\left.\frac{dy}{dx}\right|_{\pi}=\frac{\cancel{\sin\pi}\to 0+\pi\cos\pi}{\cos\pi-\cancel{\pi\sin\pi}\to 0}=\frac{\pi(-1)}{-1}=\boxed{\pi}$$

2. Find the equation of the tangent line to $r=4\cos(3\theta)$ at $\theta=\frac{\pi}{6}$ $\frac{d}{d\theta}[r=4\cos(3\theta)]$

$r'=-12\sin(3\theta)$

$$\frac{dy}{dx}=\frac{(-12\sin(3\theta))\sin\theta+4\cos(3\theta)\cos\theta}{(-12\sin(3\theta))\cos\theta-4\cos(3\theta)\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\frac{\pi}{6}}=\frac{-12\sin(\frac{\pi}{2})\sin\frac{\pi}{6}+\cancel{4\cos(\frac{\pi}{2})\cos\frac{\pi}{6}}\to 0}{-12\sin(\frac{\pi}{2})\cos\frac{\pi}{6}-\cancel{4\cos(\frac{\pi}{2})\sin\frac{\pi}{6}}\to 0}$$

$$\frac{dy}{dx}=\frac{-12(1)(\frac{1}{2})}{-12(1)(\frac{\sqrt3}{2})}=\frac{-6}{-6\sqrt3}=\frac{1}{\sqrt3}$$

$\boxed{\text{Tangent Line}}$

$\boxed{1}$ Point $(0,0)$

$\boxed{2}$ Slope $m=\frac{1}{\sqrt3}$

$\boxed{y=\frac{1}{\sqrt3}x}$

$x=r\cos\theta$

$x=4\cos(3\theta)\cos\theta=4\cos\frac{\pi}{2}\cos\frac{\pi}{6}=0$

$y=r\sin\theta$

$4\cos(3\theta)\sin\theta=4\cos\frac{\pi}{2}\sin\frac{\pi}{6}=0$

3. Show that for the circle $\frac{d}{d\theta}[r=\cos\theta+\sin\theta]$ that $\frac{dy}{dx}=\frac{\cos(2\theta)+\sin(2\theta)}{\cos(2\theta)-\sin(2\theta)}$

$\boxed{\text{Double Angle Formulas}}$

$\sin(2\theta)=2\sin\theta\cos\theta$

$\cos(2\theta)=\cos^2\theta-\sin^2\theta$

$=2\cos^2\theta-1$

$=1-2\sin^2\theta$

$r'=-\sin\theta+\cos\theta$

$$\frac{dy}{dx}=\frac{(-\sin\theta+\cos\theta)\sin\theta+(\cos\theta+\sin\theta)\cos\theta}{(-\sin\theta+\cos\theta)\cos\theta-(\cos\theta+\sin\theta)\sin\theta}$$

$$\frac{dy}{dx}=\frac{-\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta+\sin\theta\cos\theta}{-\sin\theta\cos\theta+\cos^2\theta-\sin\theta\cos\theta-\sin^2\theta}$$

$$\frac{dy}{dx}=\frac{\cos^2\theta-\sin^2\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta-2\sin\theta\cos\theta}=\frac{\cos(2\theta)+\sin(2\theta)}{\cos(2\theta)-\sin(2\theta)}$$

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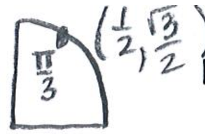
Supplement: Derivatives of Polar Functions



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4. Find the slope of the curve $\frac{d}{d\theta}\left[r = 3\cos(3\theta)\right]$ at $\theta = \frac{\pi}{3}$. $r' = -9\sin(3\theta)$

$$\frac{dy}{dx} = \frac{[-9\sin(3\theta)]\sin\theta + [3\cos(3\theta)]\cos\theta}{[-9\sin(3\theta)]\cos\theta - [3\cos(3\theta)]\sin\theta}$$

$$= \frac{\cancel{-9\sin(\pi)\sin(\tfrac{\pi}{3})}^{\,0} + 3\cos(\pi)\cos\pi/3}{\cancel{-9\sin(\pi)\cos(\pi/3)}^{\,0} - 3\cos(\pi)\sin\pi/3} = \frac{\cancel{3}(-1)(\frac{1}{2})}{-\cancel{3}(-1)(\sqrt{3}/2)} = \frac{-\frac{1}{\cancel{2}}\cdot\frac{2}{\sqrt{3}}}{\frac{\sqrt{3}}{\cancel{2}}\cdot\frac{2}{\sqrt{3}}} = \boxed{-\frac{1}{\sqrt{3}}}$$

5. Find an equation in rectangular coordinates of the line tangent to the polar-defined curve $r = \cos\theta + \sin\theta$ at $\theta = \frac{\pi}{4}$. $r = \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \sqrt{2}$ $\left(\sqrt{2}, \frac{\pi}{4}\right)$

Problem # 2 found dy/dx

$$\frac{dy}{dx} = \frac{\cos(2\theta) + \sin(2\theta)}{\cos(2\theta) - \sin(2\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\frac{\pi}{4}} = \frac{\cancel{\cos(\pi/2)}^{\,0} + \sin(\pi/2)}{\cancel{\cos(\pi/2)}^{\,0} - \sin(\pi/2)} = \frac{1}{-1} = \boxed{-1 = m}$$

$x = r\cos\theta = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 1$

$y = r\sin\theta = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 1$

$\boxed{y - 1 = 1(x - 1)}$

6. Find the point(s), in rectangular coordinates, on the curve of $\frac{d}{d\theta}\left[r = 2\sin\theta\right]$ where the tangent line is horizontal or vertical. Hint: The domain of r is $0 \le \theta \le \pi$. $r' = 2\cos\theta$

$$\frac{dy}{dx} = \frac{(2\cos\theta)\sin\theta + (2\sin\theta)\cos\theta}{(2\cos\theta)\cos\theta - (2\sin\theta)\sin\theta} = \frac{4\sin\theta\cos\theta}{2\cos^2\theta - 2\sin^2\theta}$$

$\boxed{\text{Horizontal Tangents}}$

$4\sin\theta\cos\theta = 0$

$\sin\theta = 0$ $\cos\theta = 0$

$\theta = 0, \pi$ $\theta = \frac{\pi}{2}$

(r, θ)	(x, y)
$(0, 0)$	$(0, 0)$
$(0, \pi)$	$(0, 0)$
$(2, \frac{\pi}{2})$	$(0, 2)$

$\boxed{\text{Vertical Tangents}}$

$2\cos^2\theta - 2\sin^2\theta = 0$

$2(1 - \sin^2\theta) - 2\sin^2\theta = 0$

$2 - 2\sin^2\theta - 2\sin^2\theta = 0$

$2 = 4\sin^2\theta$

$\sin^2\theta = \pm\sqrt{\frac{1}{2}}$ $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$\sin\theta = \pm\frac{\sqrt{2}}{2}$

(r, θ)	(x, y)
$(\sqrt{2}, \frac{\pi}{4})$	$(1, 1)$
$(\sqrt{2}, \frac{3\pi}{4})$	$(-1, 1)$

7. Find the point(s), in rectangular coordinates, on the curve of $\frac{d}{d\theta}\left[r = 1 + \cos\theta\right]$ where the tangent line is horizontal or vertical. Hint: The domain of r is $0 \le \theta \le 2\pi$. $r' = -\sin\theta$

$$\frac{dy}{dx} = \frac{(-\sin\theta)\sin\theta + (1 + \cos\theta)\cos\theta}{(-\sin\theta)\cos\theta - (1 + \cos\theta)\sin\theta}$$

$$\frac{dy}{dx} = \frac{-\sin^2\theta + \cos\theta + \cos^2\theta}{-\sin\theta\cos\theta - \sin\theta - \sin\theta\cos\theta}$$

$$\frac{dy}{dx} = \frac{-\sin^2\theta + \cos\theta + \cos^2\theta}{-\sin\theta - 2\sin\theta\cos\theta}$$

$\boxed{\text{Vertical}}$ $-\sin\theta[1 + 2\cos\theta]$

$\sin\theta = 0$ $\cos\theta = -\frac{1}{2}$

$\theta = 0, \pi, 2\pi$ $\theta = +2\pi/3, 4\pi/3$

(r, θ)	(x, y)
$\boxed{(2, 0)}$	$\boxed{(2, 0)}$
~~$(0, \pi)$~~	~~$(0, 0)$~~
$(\frac{1}{2}, 2\pi/3)$	$(-\frac{1}{4}, \frac{\sqrt{3}}{4})$
$(\frac{1}{2}, 4\pi/3)$	$(-\frac{1}{4}, -\frac{\sqrt{3}}{4})$

$\boxed{\text{Horizontal Tangents}}$

$-\sin^2\theta + \cos\theta + \cos^2\theta = 0$

$-(1 - \cos^2\theta) + \cos\theta + \cos^2\theta = 0$

$-1 + \cos^2\theta + \cos\theta + \cos^2\theta = 0$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1)$

$\cos\theta = 1/2$ $\cos\theta = -1$

$\theta = \pi/3, 5\pi/3$ $\theta = \pi$

(r, θ)	(x, y)
$(3/2, \pi/3)$	$(3/4, 3\sqrt{3}/4)$
$(3/2, 5\pi/3)$	$(3/4, -3\sqrt{3}/4)$
~~$(0, \pi)$~~	~~$(0, 0)$~~

$$\left(\overset{r}{2}, \overset{\theta}{0}\right) \quad r = 1 + \cos(0) = 1 + 1$$

$$(0, \pi) \quad r = 1 + \cos\pi = 1 - 1$$

$$\left(\tfrac{1}{2}, \tfrac{2\pi}{3}\right) \quad r = 1 + \cos\tfrac{2\pi}{3} = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

$$\left(\tfrac{1}{2}, \tfrac{4\pi}{3}\right) \quad r = 1 + \cos\tfrac{4\pi}{3} = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

$$\left(\overset{x}{2}, \overset{y}{0}\right) \quad \begin{matrix} 2\cos(0) = 2(1) \\ 2\sin(0) = 2(0) \end{matrix}$$

$$(0, 0) \quad \begin{matrix} 0\cos\pi \\ 0\sin\pi \end{matrix}$$

$$\left(\tfrac{-1}{4}, \tfrac{\sqrt{3}}{4}\right) \quad \begin{matrix} \tfrac{1}{2}\cos\tfrac{2\pi}{3} = \tfrac{1}{2}\left(-\tfrac{1}{2}\right) \\ \tfrac{1}{2}\sin\tfrac{2\pi}{3} = \tfrac{1}{2}\left(\tfrac{\sqrt{3}}{2}\right) \end{matrix}$$

$$\left(\tfrac{-1}{4}, \tfrac{-\sqrt{3}}{2}\right) \quad \begin{matrix} \tfrac{1}{2}\cos\tfrac{4\pi}{3} = \tfrac{1}{2}\left(-\tfrac{1}{2}\right) \\ \tfrac{1}{2}\sin\tfrac{4\pi}{3} \quad \tfrac{1}{2}\left(-\tfrac{\sqrt{3}}{2}\right) \end{matrix}$$

$$r = 1 + \cos\theta$$

$$(r, \theta)$$

$$\left(\tfrac{3}{2}, \pi/3\right) \quad r = 1 + \cos\tfrac{\pi}{3} = 1 + \tfrac{1}{2} = \tfrac{3}{2}$$

$$\left(\tfrac{3}{2}, \tfrac{5\pi}{3}\right)$$

$$(0\ \pi) \quad r = 1 + \cos\pi = 1 - 1 = 0$$

$$x, y$$

$$\left(\tfrac{3}{4}, \tfrac{3\sqrt{3}}{4}\right) \quad \begin{matrix} 3/2\cos\tfrac{\pi}{3} \quad \tfrac{3}{2}\left(\tfrac{1}{2}\right) \\ 3/2\sin\pi/3 \quad \tfrac{3}{2}\tfrac{\sqrt{3}}{2} \end{matrix}$$

$$\left(\tfrac{3}{4}, \tfrac{-3\sqrt{3}}{4}\right) \quad \begin{matrix} 3/2\cos 5\pi/3 \\ 3/2\sin 5\pi/3 \end{matrix}$$

$$(0, 0) \quad \begin{matrix} 0\cos\pi \\ 0\sin\pi \end{matrix}$$