

Supplement Derivatives of Polar Functions

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{r'(\theta) \sin \theta + r \cos \theta}{r'(\theta) \cos \theta - r \sin \theta}$$

1. Find the slope of the tangent line to $r = \theta$ at $\theta = \frac{\pi}{2}$ and $\theta = \pi$

$$\frac{dy}{dx} = \frac{(1)\sin\theta + \theta \cos\theta}{(1)\cos\theta - \theta \sin\theta}$$

$$\frac{dy}{dx} \Big|_{\frac{\pi}{2}} = \frac{\sin\frac{\pi}{2} + \frac{\pi}{2}\cos\frac{\pi}{2}}{\cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2}} = \frac{1}{-\frac{\pi}{2}} = \boxed{-\frac{2}{\pi}}$$

$$\begin{aligned} \frac{dy}{dx} \Big|_{\pi} &= \frac{\sin\pi + \pi \cos\pi}{\cos\pi - \pi \sin\pi} \\ &= \frac{0 - \pi(-1)}{-1} = \boxed{\pi} \end{aligned}$$

2. Find the equation of the tangent line to $r = 4\cos(3\theta)$ at $\theta = \frac{\pi}{6}$

$$\frac{dy}{dx} = \frac{(-12\sin(3\theta))\sin\theta + 4\cos(3\theta)\cos\theta}{(-12\sin(3\theta))\cos\theta - 4\cos(3\theta)\sin\theta}$$

$$\begin{aligned} \frac{dy}{dx} \Big|_{\frac{\pi}{6}} &= \frac{-12\sin\left(\frac{\pi}{2}\right)\sin\frac{\pi}{6} + 4\cos\left(\frac{\pi}{2}\right)\cos\frac{\pi}{6}}{-12\sin\left(\frac{\pi}{2}\right)\cos\frac{\pi}{6} - 4\cos\left(\frac{\pi}{2}\right)\sin\frac{\pi}{6}} \\ &\rightarrow 0 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-12(1)\left(\frac{1}{2}\right)}{-12(1)\left(\frac{\sqrt{3}}{2}\right)} = \frac{-6}{-6\sqrt{3}} = \frac{1}{\sqrt{3}} & x = r\cos\theta & y = \frac{1}{\sqrt{3}}x \\ &\quad x = 4\cos(3\theta)\cos\theta = 4\cos\frac{\pi}{2}\cos\frac{\pi}{6} = 0 \\ &\quad y = 4\cos(3\theta)\sin\theta = 4\cos\frac{\pi}{2}\sin\frac{\pi}{6} = 0 \end{aligned}$$

3. Show that for the circle $r = \cos\theta + \sin\theta$ that $\frac{dy}{dx} = \frac{\cos(2\theta) + \sin(2\theta)}{\cos(2\theta) - \sin(2\theta)}$

$$r' = -\sin\theta + \cos\theta$$

$$\frac{dy}{dx} = \frac{(-\sin\theta + \cos\theta)\sin\theta + (\cos\theta + \sin\theta)\cos\theta}{(-\sin\theta + \cos\theta)\cos\theta - (\cos\theta + \sin\theta)\sin\theta}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta + \sin\theta\cos\theta}{-\sin\theta\cos\theta + \cos^2\theta - \sin\theta\cos\theta - \sin^2\theta} \\ &\quad \cancel{-\sin^2\theta} \cancel{+ \cos^2\theta} \cancel{- \sin\theta\cos\theta} \cancel{- \sin^2\theta} \end{aligned}$$

$$\frac{dy}{dx} = \frac{\cos^2\theta - \sin^2\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta} = \frac{\cos(2\theta) + \sin(2\theta)}{\cos(2\theta) - \sin(2\theta)}$$

Double Angle Formulas

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}$$



4. Find the slope of the curve $r = 3\cos(3\theta)$ at $\theta = \frac{\pi}{3}$. $r' = -9\sin(3\theta)$

$$\frac{dy}{dx} = \frac{[-9\sin(3\theta)]\sin\theta + [3\cos(3\theta)]\cos\theta}{[-9\sin(3\theta)]\cos\theta - [3\cos(3\theta)]\sin\theta}$$

$$= \frac{-9\sin(\pi)\sin(\frac{\pi}{3}) + 3\cos(\pi)\cos(\frac{\pi}{3})}{-9\sin(\pi)\cos(\frac{\pi}{3}) - 3\cos(\pi)\sin(\frac{\pi}{3})} = \frac{3(-1)(\frac{1}{2})}{-3(-1)(\frac{\sqrt{3}}{2})} = \frac{-\frac{3}{2}}{\frac{3\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

5. Find an equation in rectangular coordinates of the line tangent to the polar-defined curve

$$r = \cos\theta + \sin\theta \text{ at } \theta = \frac{\pi}{4}. \quad r = \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2} \quad (\sqrt{2}, \frac{\pi}{4})$$

Problem #2 found $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\cos(2\theta) + \sin(2\theta)}{\cos(2\theta) - \sin(2\theta)}$$

$$\left. \frac{dy}{dx} \right|_{\frac{\pi}{4}} = \frac{\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})}{\cos(\frac{\pi}{2}) - \sin(\frac{\pi}{2})} = \frac{1}{-1} = -1 = m$$

$$x = r \cos\theta = \sqrt{2} \cos\frac{\pi}{4} = \sqrt{2}(\frac{\sqrt{2}}{2}) = 1$$

$$y = r \sin\theta = \sqrt{2} \sin\frac{\pi}{4} = \sqrt{2}(\frac{\sqrt{2}}{2}) = 1$$

$$y - 1 = 1(x - 1)$$

6. Find the point(s), in rectangular coordinates, on the curve of $r = 2\sin\theta$ where the tangent line is horizontal or vertical. Hint: The domain of r is $0 \leq \theta \leq \pi$. $r' = 2\cos\theta$

$$\frac{dy}{dx} = \frac{(2\cos\theta)\sin\theta + (2\sin\theta)\cos\theta}{(2\cos\theta)\cos\theta - (2\sin\theta)\sin\theta} = \frac{4\sin\theta\cos\theta}{2\cos^2\theta - 2\sin^2\theta}$$

$$\begin{aligned} \sin^2\theta &= \frac{1}{2} & \theta &= \frac{\pi}{4}, \frac{3\pi}{4} \\ \sin\theta &= \pm\frac{\sqrt{2}}{2} \end{aligned}$$

Horizontal Tangents (r, θ) (x, y)

$$4\sin\theta\cos\theta = 0 \quad (0, 0) \quad (0, 0)$$

$$\sin\theta = 0 \quad \cos\theta = 0 \quad (0, \pi) \quad (0, 0)$$

$$\theta = 0, \pi \quad \theta = \frac{\pi}{2} \quad (2, \frac{\pi}{2}) \quad (0, 2)$$

Vertical Tangents

$$2\cos^2\theta - 2\sin^2\theta = 0$$

$$2(1 - \sin^2\theta) - 2\sin^2\theta = 0$$

$$2 - 2\sin^2\theta - 2\sin^2\theta = 0$$

$$2 = 4\sin^2\theta$$

$$(r, \theta) \quad (x, y)$$

$$(\sqrt{2}, \frac{\pi}{4}) \quad (1, 1)$$

$$(\sqrt{2}, \frac{3\pi}{4}) \quad (-1, 1)$$

7. Find the point(s), in rectangular coordinates, on the curve of $r = 1 + \cos\theta$ where the tangent line is horizontal or vertical. Hint: The domain of r is $0 \leq \theta \leq 2\pi$. $r' = -\sin\theta$

$$\frac{dy}{dx} = \frac{(-\sin\theta)\sin\theta + (1 + \cos\theta)\cos\theta}{(-\sin\theta)\cos\theta - (1 + \cos\theta)\sin\theta}$$

Horizontal Tangents

$$-\sin^2\theta + \cos\theta + \cos^2\theta = 0$$

$$-(1 - \cos^2\theta) + \cos\theta + \cos^2\theta = 0$$

$$-1 + \cos^2\theta + \cos\theta + \cos^2\theta = 0$$

$$2\cos^2\theta + \cos\theta - 1 = 0$$

$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$$\cos\theta = \frac{1}{2} \quad \cos\theta = -1$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$(r, \theta) \quad (x, y)$$

$$(\frac{1}{2}, \frac{\pi}{3}) \quad (-\frac{1}{4}, \frac{3}{4})$$

$$(\frac{1}{2}, \frac{4\pi}{3}) \quad (-\frac{1}{4}, -\frac{3}{4})$$

$$(\frac{1}{2}, \frac{5\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{7\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{11\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{13\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{17\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{19\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{21\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{23\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{25\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{27\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{29\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{31\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{33\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{35\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{37\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{39\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{41\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{43\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{45\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{47\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{49\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{51\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{53\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{55\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{57\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{59\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{61\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{63\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{65\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{67\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{69\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{71\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{73\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{75\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{77\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{79\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{81\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{83\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{85\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{87\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{89\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{91\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{93\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{95\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{97\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{99\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{101\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{103\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{105\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{107\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{109\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{111\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{113\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{115\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{117\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{119\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{121\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{123\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{125\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{127\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{129\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{131\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{133\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{135\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{137\pi}{3}) \quad (0, 0)$$

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$$(\frac{1}{2}, \frac{141\pi}{3}) \quad (0, 0)$$

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$$(\frac{1}{2}, \frac{147\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{149\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{151\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{153\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{155\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{157\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{159\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{161\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{163\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{165\pi}{3}) \quad (0, 0)$$

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$$(\frac{1}{2}, \frac{169\pi}{3}) \quad (0, 0)$$

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$$(\frac{1}{2}, \frac{205\pi}{3}) \quad (0, 0)$$

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$$(\frac{1}{2}, \frac{247\pi}{3}) \quad (0, 0)$$

$$(\frac{1}{2}, \frac{249\pi}{3}) \quad (0, 0)$$

$$\begin{array}{ll}
 (\frac{1}{2}, 0) \quad r = 1 + \cos(0) = 1+1 & (\frac{x}{2}, \frac{y}{2}) \quad \begin{array}{l} 2\cos(0) = 2(1) \\ 2\sin(0) = 2(0) \end{array} \\
 (0, \pi) \quad r = 1 + \cos(\pi) = 1-1 & (0, 0) \quad \begin{array}{l} 0\cos\pi \\ 0\sin\pi \end{array} \\
 (\frac{1}{2}, \frac{2\pi}{3}) \quad r = 1 + \cos\frac{2\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2} & (-\frac{1}{4}, \frac{\sqrt{3}}{4}) \quad \begin{array}{l} \frac{1}{2}\cos\frac{2\pi}{3} = \frac{1}{2}(-\frac{1}{2}) \\ \frac{1}{2}\sin\frac{2\pi}{3} = \frac{1}{2}(\frac{\sqrt{3}}{2}) \end{array} \\
 (\frac{1}{2}, \frac{4\pi}{3}) \quad r = 1 + \cos\frac{4\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2} & (-\frac{1}{4}, -\frac{\sqrt{3}}{2}) \quad \begin{array}{l} \frac{1}{2}\cos\frac{4\pi}{3} = \frac{1}{2}(-\frac{1}{2}) \\ \frac{1}{2}\sin\frac{4\pi}{3} = \frac{1}{2}(-\frac{\sqrt{3}}{2}) \end{array}
 \end{array}$$

$$r = 1 + \cos\theta$$

$$\begin{array}{ll}
 (r, \theta) & x, y \\
 (\frac{3}{2}, \frac{\pi}{3}) \quad r = 1 + \cos\frac{\pi}{3} = 1 + \frac{1}{2} = \frac{3}{2} & (\frac{3}{4}, \frac{3\sqrt{3}}{4}) \quad \begin{array}{l} \frac{3}{2}\cos\frac{\pi}{3} \\ \frac{3}{2}\sin\frac{\pi}{3} \end{array} \\
 (\frac{3}{2}, \frac{5\pi}{3}) & (\frac{3}{4}, -\frac{3\sqrt{3}}{4}) \quad \begin{array}{l} \frac{3}{2}\cos\frac{5\pi}{3} \\ \frac{3}{2}\sin\frac{5\pi}{3} \end{array} \\
 (0, \pi) \quad r = 1 + \cos\pi = 1-1=0 & (0, 0) \quad \begin{array}{l} 0\cos\pi \\ 0\sin\pi \end{array}
 \end{array}$$