Day 2 ☺

Notes: Partial Fraction & Long Division (Rational Functions)

When integrating you always try and use u-substitution first. If that does not work then:

1. If degree of the numerator ≥ degree of denominator, then use long division to simplify the integrand. Then integrate. (Marilyn or Dolly)
2. If the degree of the numerator < degree of denominator, then use partial fraction decomposition to simplify the integrand. Then integrate. (J-Lo)

Example One: $\int \frac{dx}{x^2-7x+10}$ J-Lo

$$\int \frac{dx}{x^2-7x+10} = \int \frac{\frac{1}{3}}{x-5} + \int \frac{-\frac{1}{3}}{x-2}$$

$$= \frac{1}{3}\int \frac{1}{x-5} - \frac{1}{3}\int \frac{1}{x-2}$$

$u = x-5$ $du = dx$ $u = x-2$ $du = dx$

$$= \frac{1}{3}\int \frac{1}{u}du - \frac{1}{3}\int \frac{1}{u}du$$

$$\boxed{= \frac{1}{3}\ln|x-5| - \frac{1}{3}\ln|x-2| + C}$$

Partial Fraction Decomposition

$\boxed{1.}$ Rewrite w/ factored bottom $\frac{1}{(x-5)(x-2)}$

$\boxed{2}$ Rewrite & get common denominator $\frac{1}{(x-5)(x-2)} = \frac{A(x-2)}{(x-5)(x-2)} + \frac{B(x-5)}{(x-2)(x-5)}$

$\boxed{3}$ Rewrite w/out denominator $1 = A(x-2) + B(x-5)$

$\boxed{4}$ Solve for A & B

$\boxed{\text{let } x=2}$
$1 = A(0) + B(-3)$
$1 = -3B$
$B = -\frac{1}{3}$

$\boxed{\text{let } x=5}$
$1 = A(3) + B(0)$
$1 = 3A$
$A = \frac{1}{3}$

Example Two: $\int \frac{x^2+2}{(x-1)(2x-8)(x+2)}dx$ J-Lo

$$\int \frac{-\frac{1}{6}}{x-1} + \frac{1}{2}\int \frac{2 \cdot 1}{2x-8} + \int \frac{\frac{1}{6}}{x+2}$$

$u = x-1$ $du = dx$ $u = 2x-8$ $du = 2dx$ $u = x+2$ $du = dx$

$$-\frac{1}{6}\int \frac{1}{u} + \frac{1}{2}\int \frac{1}{u} + \frac{1}{6}\int \frac{1}{u}$$

$$-\frac{1}{6}\ln|x-1| + \frac{1}{2}\ln|2x-8| + \frac{1}{6}\ln|x+2| + C$$

$$\frac{x^2+2}{(x-1)(2x-8)(x+2)} = \frac{A(2x-8)(x+2)}{(x-1)(2x-8)(x+2)} + \frac{B(x-1)(x+2)}{(2x-8)(x-1)(x+2)} + \frac{C(x-1)(2x-8)}{(x+2)(x-1)(2x-8)}$$

$$x^2+2 = A(2x-8)(x+2) + B(x-1)(x+2) + C(x-1)(2x-8)$$

$\boxed{\text{let } x=4}$
$(4)^2+2 = A(0)(6) + B(3)(6) + C(3)(0)$
$18 = 18B$ $B = 1$

$\boxed{\text{let } x=-2}$
$(-2)^2+2 = A(-12)(0) + B(-3)(0) + C(-3)(-12)$
$6 = 36C$ $C = 1/6$

$\boxed{\text{let } x=1}$
$(1)^2+2 = A(-6)(3) + B(0)(3) + C(0)(-6)$
$3 = -18A$ $A = -1/6$

Example Three: $\int \frac{x^3+1}{x^2-4}\,dx$ Dolly

$$\int x + \frac{4x+1}{x^2-4}$$

$$\int x + \int \frac{4x+1}{x^2-4}$$

$$\int x + \int \frac{9/4}{x-2} + \int \frac{\frac{7}{4}}{x+2}$$

$$\frac{x^2}{2} + \frac{9}{4}\ln|x-2| + \frac{7}{4}\ln|x+2| + C$$

Traditional Division

$$\begin{array}{r} x \\ x^2-4\overline{)\,x^3+0x^2+0x+1} \\ -x^3 \qquad +4x \\ \hline 4x+1 \end{array}$$

- Divide $\frac{x^3}{x^2}$
- Mult.
- Subtract

Partial Fraction Decomp

$$\frac{4x+1}{(x-2)(x+2)} = \frac{A^{(x+2)}}{\underset{(x+2)}{(x-2)}} + \frac{B(x-2)}{(x+2)(x-2)}$$

$$4x+1 = A(x+2) + B(x-2)$$

$\boxed{\text{let } x=-2}$

$4(-2)+1 = A(0) + B(-4)$

$-7 = -4B$

$B = 7/4$

$\boxed{\text{let } x=2}$

$4(2)+1 = A(4) + B(0)$

$9 = 4A$

$A = 9/4$

Example Four: $\int \frac{3x-9}{(x-1)(x+2)^2}\,dx$ J-Lo

$$\int \frac{-2/3}{x-1} + \int \frac{2/3}{x+2} + \int \frac{5}{(x+2)^2}$$

$$-\frac{2}{3}\int \frac{1}{u} + \frac{2}{3}\int \frac{1}{u} + 5\int \frac{1}{u^2}$$

$$-\frac{2}{3}\int \frac{1}{u} + \frac{2}{3}\int \frac{1}{u} + 5\int u^{-2}$$

$$-\frac{2}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| + \frac{5u^{-1}}{-1} + C$$

$$-\frac{2}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| - \frac{5}{x+2} + C$$

Repeated Linear Factors

$$\frac{3x-9}{(x-1)(x+2)^2} = \frac{A^{(x+2)^2}}{\underset{(x+2)^2}{x-1}} + \frac{B^{(x-1)(x+2)}}{\underset{(x-1)(x+2)}{x+2}} + \frac{C\,(x-1)}{(x+2)^2(x-1)}$$

$$3x-9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

$\boxed{\text{let } x=-2}$

$3(-2)-9 = -3C$

$-15 = -3C$

$C = 5$

$\boxed{\text{let } x=1}$

$3(1)-9 = A(3)^2$

$-6 = 9A$

$A = -2/3$

$\boxed{\text{let } C=5,\ A=-2/3,\ x=0}$ $B = 2/3$

$3(0)-9 = -2/3(2)^2 + B(-1)(2) + 5(-1)$

$\underset{+5}{-9} = -8/3 - 2B \cancel{-5}$

$\underset{+8/3}{-4} = -\cancel{8}/3 - 2B$

$\left(-\frac{1}{2}\right) -\frac{4}{3} = -\cancel{2}B\left(-\frac{1}{\cancel{2}}\right)$

Lets Try More

1. $\int_0^3 \frac{5x^3-x}{x^2+1}$ Dolly

$$\int_0^3 5x + \int_0^3 \frac{-6x}{x^2+1}$$

$$\int_0^3 5x - 6\cdot\frac{1}{2}\int_0^3 \frac{2x}{x^2+1} \qquad u=x^2+1 \quad du=2x\,dx$$

$$\int_0^3 5x - 3\int_0^3 \frac{1}{u}\,du$$

$$\cancel{\int_0^3}\ \frac{5x^2}{2} - 3\ln|x^2+1|\Big]_0^3$$

$$\frac{5(9)}{2} - 3\ln(10) - \underset{0}{\cancel{\frac{5(0)}{2}}} + \underset{0}{\cancel{3\ln(1)}} = \frac{45}{2} - 3\ln(10)$$

Long Division

$$\begin{array}{r} 5x \qquad\qquad\quad \\ x^2+1 \enclose{longdiv}{\cancel{5x^3}+0x^2-x+0} \\ \underline{-\cancel{5x^3} \qquad +5x} \\ -6x \end{array} \qquad \frac{5x^3}{x^2}=5x$$

2. $\int_2^3 \frac{dt}{t^2+3t-4}$

$$\frac{1}{(t-1)(t+4)} = \frac{A^{(t+4)}}{(t-1)_{(t+4)}} + \frac{B^{(t-1)}}{(t+4)_{(t-1)}}$$

$$1 = A(t+4)+B(t-1)$$

$\boxed{\text{let } t=-4}$

$1=-5B$

$B=\frac{-1}{5}$

$\boxed{\text{let } t=1}$

$1=5A$

$A=\frac{1}{5}$

$$\int_2^3 \frac{\frac{1}{5}}{t-1} + \int_2^3 \frac{-\frac{1}{5}}{t+4}$$

$$\frac{1}{5}\ln|t-1| - \frac{1}{5}\ln|t+4|\Big]_2^3$$

$$\frac{1}{5}\ln(2) - \frac{1}{5}\ln(7) - \underset{0}{\cancel{\frac{1}{5}\ln(1)}} + \frac{1}{5}\ln(6)$$

$$\frac{1}{5}\ln(2) + \frac{1}{5}\ln(6) - \frac{1}{5}\ln(7)$$

$$\frac{1}{5}\left[\ln(2)+\ln(6)-\ln(7)\right]$$

$$\frac{1}{5}\ln\left(\frac{2\cdot 6}{7}\right) = \frac{1}{5}\ln\left(\frac{12}{7}\right)$$

3. $\int \frac{u^4}{u^2+3} du$ (Dolly)

$$\begin{array}{r} u^2-3 \\ u^2+3 \overline{\big)\, \cancel{u^4}+0u^3+0u^2+0u+0} \\ -u^4 \quad \mp 3u^2 \\ \hline -3u^2+0u+0 \\ +3u^2 \quad +9 \\ \hline 9 \end{array}$$

$\frac{u^4}{u^2} = u^2$

$\frac{-3u^2}{u^2} = -3$

$$\int u^2-3+\frac{9}{u^2+3}\,du$$

$$\int u^2-3+9\int\frac{1}{u^2+3}$$

$$\int u^2-3+9\int\frac{1}{3\left(\frac{u^2}{3}+1\right)}$$

$$\int u^2-3+9\left(\frac{1}{3}\right)\sqrt{3}\int\frac{1\,{}^{1}\!/\!\sqrt{3}}{\left(\frac{u}{\sqrt{3}}\right)^2+1}$$

$w=\frac{1}{\sqrt{3}}u$

$dw=\frac{1}{\sqrt{3}}du$

$$\int u^2-3+3\sqrt{3}\int\frac{1}{u^2+1} = \frac{u^3}{3}-3u+3\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}u\right)+C$$

4. $\int \frac{3x-4}{x^2-5x-6} dx$

$$\frac{3x-4}{(x+1)(x-6)} = \frac{A(x-6)}{x+1\,(x-6)} + \frac{B(x+1)}{x-6\,(x+1)}$$

$$\int\frac{1}{x+1}+\int\frac{2}{x-6}$$

$$\ln|x+1|+2\ln|x-6|+C$$

$3x-4 = A(x-6)+B(x+1)$

$\boxed{\text{let } x=6}$ $3(6)-4=7B$ $14=7B$ $B=2$

$\boxed{\text{let } x=-1}$ $3(-1)-4=-7A$ $-7=-7A$ $A=1$