Notes: Logistic Equation

Additional Differential Equations Day 3

$\dot{y}=$ It is notation used in physics to represent a derivative with respect to time.

$\dot{y}=$ dy/dt

Logistic Differential Equation: Predicts that populations grow exponentially. It takes into account limitations like food scarcity, competition between species, and environmental limitations. It is these limitations that make it different than exponential growth and decay that we learned last semester.

Exponential Growth/Decay	Newton's Law of Cooling	Logistic Equation
$\frac{dy}{dt}=ky$	$\frac{dy}{dt}=-k(y-T_0)$	$\frac{dy}{dt}=ky\left(1-\frac{1y}{A}\right)$
$y=P_0e^{kt}$	$y=Ce^{-kt}+T_0$	$y=\frac{A}{1-\frac{e^{-kt}}{C}}$
K= growth OR decay rate P_0= initial amount	K= cooling rate T_0= "outside temperature" C= Initial temp $-T_0$	K= growth rate A= carrying capacity $C=\frac{y_0}{y_0-A}$ where y_0= initial y-value

Example One: $\dot{y}=.3y(4-y)$ and $y(0)=1$. What does $y=$? What is the growth constant and the carrying capacity for this equation?

$\frac{dy}{dt}=.3y[4-y]$

$\frac{dy}{dt}=.3y(4)\left[\frac{4}{4}-\frac{y}{4}\right]$

$\frac{dy}{dt}=1.2y\left[1-\frac{1y}{4}\right]$

$A=4 \quad K=1.2$

$C=\frac{y_0}{y_0-A}$

$C=\frac{1}{1-4}$

$C=-\frac{1}{3}$

$y=\frac{A}{1-\frac{e^{-kt}}{C}}$

$y=\frac{4}{1+\frac{e^{-1.2t}\cdot\frac{3}{1}}{+\frac{1}{3}\cdot\frac{3}{1}}}$

$y=\frac{4}{1+3e^{-1.2t}}$

$\frac{.01}{.25} = \frac{1}{25}$ $C = \frac{y_0}{y_0 - A}$ $y = \frac{A}{1 - \frac{e^{-kt}}{C}}$

Example Two: $\dot{y} = y(\frac{1}{4} - .01y)$ and $y(0) = 5$. What does $y =$? What is the growth constant and the carrying capacity for this equation?

$\frac{dy}{dt} = y[.25 - .01y]$ $A = 25$ $y = \frac{25}{1 + \frac{e^{-.25t}\cdot\frac{4}{1}}{+\frac{1}{4}\cdot\frac{4}{1}}}$ $y = \frac{25}{1 + 4e^{-.25t}}$

$\frac{dy}{dt} = y(.25)\left[\frac{.25}{.25} - \frac{.01y}{.25}\right]$ $K = .25$

$C = \frac{5}{5 - 25}$

$\frac{dy}{dt} = .25y\left[1 - \frac{1}{25}y\right]$ $C = -\frac{1}{4}$

Example Three: $\dot{y} = .003y(10 - 3y)$ and $y(0) = .25$. What does $y =$? What is the growth constant and the carrying capacity for this equation?

$\frac{dy}{dt} = .003y(10)\left[\frac{10}{10} - \frac{3y}{10}\right]$ $A = \frac{10}{3}$ $C = \frac{\frac{1}{4}\cdot\frac{12}{-37}}{\frac{-37}{12}\cdot\frac{12}{-37}}$ $y = \frac{\frac{10}{3}}{1 + \frac{e^{-.03t}\,\frac{37}{3}}{+\frac{3}{37}\cdot\frac{37}{3}}}$

$\frac{dy}{dt} = .03y\left[1 - \frac{\frac{3}{10}y\frac{10}{3}}{1\cdot\frac{10}{3}}\right]$ $K = .03$

$\frac{dy}{dt} = .03y\left[1 - \frac{1y}{10/3}\right]$ $C = \frac{\frac{1}{4}}{\frac{3\cdot 1}{3\cdot 4} - \frac{10\cdot 4}{3\cdot 4}}$ $C = \frac{-3}{37}$ $y = \frac{\frac{10}{3}\cdot 3}{\left(1 + \frac{37e^{-.03t}}{3}\right)\cdot 3} = \frac{10}{3 + 37e^{-.03t}}$

Example Four: $\frac{dP}{dt} = .1P - .001P^2$

a.) Find $P(t)$ if $P(0) = 50$

$\frac{dP}{dt} = .1P\left[\frac{.1P}{.1P} - \frac{.001P^2}{.1P}\right]$ $\frac{dP}{dt} = .1P\left[1 - \frac{P}{100}\right]$ $C = \frac{50}{50 - 100}$ $P(t) = \frac{100}{1 + \frac{e^{-.1t}}{+1}}$

$A = 100$ $K = .1$ $C = -1$

b.) Find $P(10)$, $P(15)$, & $P(20)$

$P(t) = \frac{100}{1 + e^{-.1t}}$ $P(10) = \frac{100}{1 + e^{(-.1)(10)}} = 73$ $P(15) = 81$ $P(20) = 88$

c.) How long will it take for the population to be 75? $P(t) = 75$?

$75 = \frac{100}{1 + e^{-.1t}}$ F2 1: solve

solve(75 = 100 ÷ (1 + e^(-.1*t)), t) $t = 10.906$

d.) When is the population growing the fastest?

$\overline{dP/dt} = \max$

Your population is always growing the fastest when it is ½ the carrying capacity.

$\frac{d}{dt}\left[\frac{dP}{dt} = .1P - .001P^2\right]$

$\frac{d^2P}{dt^2} = .1 - .002P$ $0 = .1 - .002P$

$\frac{.002P}{.002} = \frac{.1}{.002}$ $P = \frac{100}{2} = 50$