Notes: Euler's Method

Additional Differential Equations
Day 2

Leonhard Euler (/ˈɔɪlər/ **OY**-*lər*); April 1707 – September 1783

AB Review: Find the tangent line to $y = \sqrt{x}$ at the point $(25,5)$. Use the tangent line to approximate $y(26)$. Find the % error.

Tangent Line

1. Point $(25,5)$
2. Slope $y = \sqrt{x} = x^{1/2}$

$y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$y'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$

$y - y_1 = m(x - x_1)$

$y - 5 = \frac{1}{10}(x - 25)$

$y = \frac{1}{10}(x-25) + 5$

$y(26) = \frac{1}{10}(26-25) + 5$

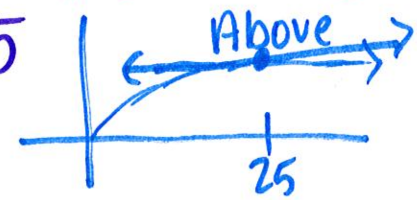
$y(26) \approx 5.1$

Approx

$\sqrt{26} \approx 5.1$

Actual

$\sqrt{26} = 5.09902$



Euler's Method is just another way to find an approximate. We are going to use the table method.

x	$(\Delta x)\left(\text{slope or } \frac{dy}{dx} \text{ or } f'(x)\right)$	y
initial x		initial y
$x + \Delta x$	(previous slope) $\cdot \Delta x$	Previous y + middle column
$x + 2\Delta x$		
$x + 3\Delta x$		

Example One: Use Eulers Method to approximate $y(26)$. Given $y = \sqrt{x}$, $f(25) = 5$, & $\Delta x = .2$. Find the % error of your approximation.

x	$\Delta x \cdot y'$	y
25	$.2\left(\frac{1}{2\sqrt{x}}\right)$	5
25.2	$.2\left(\frac{1}{2\sqrt{25}}\right) = .02$	5.02
25.4	$.2\left(\frac{1}{2\sqrt{25.2}}\right) = .01992$	5.03992
25.6	$.2\left(\frac{1}{2\sqrt{25.4}}\right) = .01984$	5.05976
25.8	$.2\left(1/2\sqrt{25.6}\right) = .01976$	5.07953
26	$.2\left(1/2\sqrt{25.8}\right) = .01968$	5.09921

$y = \sqrt{x} = x^{1/2}$

$y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Approx

$\sqrt{26} \approx 5.09921$

Actual ≈ 5.09902

Example Two: Use Euler's Method starting at $(1,2)$ with $\Delta x = .2$ to estimate $y(2)$ if $y' = \frac{y}{x}$.

x	$\Delta x \cdot y'$	y
1	$(.2)\, y/x$	2
1.2	$(.2)\, 2/1 = .4$	2.4
1.4	$(.2)\, 2.4/1.2 = .4$	2.8
1.6	$(.2)\, 2.8/1.4 = .4$	3.2
1.8	$(.2)\, 3.2/1.6 = .4$	3.6
2	$(.2)\, 3.6/1.8 = .4$	4

$y(2) \approx 4$

Example Three: Use Euler's Method starting at $\boxed{\overset{x\ y}{(0,3)}}$ with $\Delta x = .5$ to estimate $y(2)$ given $y' = 2xy + 2y$. Find the actual solution to the differential equation using the initial value $(0,3)$. Is your approximation to $y(2)$ using Euler's method an over estimate or and under estimate? Why?

x	$\Delta x \cdot y'$	y
0	$\frac{1}{2}[2y(x+1)]$	3
.5	$3(0+1) = 3$	6
1	$6(.5+1) = 6(3/2) = 9$	15
1.5	$15(1+1) = 30$	45
2	$45(1.5+1) = 112.5$	157.5

$\frac{dy}{dx} = 2y(x+1)$

$\int \frac{1}{y}\,dy = \int 2x+2\,dx$

$\ln|y| = x^2+2x+C$

$\ln 3 = 0^2+2(0)+C$

$C = \ln 3$

$\ln|y| = x^2+2x+\ln 3$

$|y| = 3e^{x^2+2x}$

$y = 3e^{x^2+2x}$

$y(2) = 8942$

Example Four: Suppose a continuous function f and its derivative f' have values that are given in the following table. Given that f(0)=2, use Euler's method with two steps of equal size to approximate the value of f(-2).

x	0	-1	-2
f'(x)	.5	1	3
f(x)	2	1.5	.5

x	$\Delta x \cdot f'(x)$	y
0	$-1 \cdot f'(x)$	2
-1	$-1 \cdot f'(0) = -1(.5) = -.5$	1.5
-2	$-1 \cdot f'(-1)$ $-1(1) = -1$	.5

$f(-2) \approx .5$