Notes: Day 1

AP Calculus
Wksht. Differential Equations: Additional Topics

Name__________
Date__________ Period__________

Consider the differential equation: $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$.

a.) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \frac{1}{2}$.

$\int e^{2y}\,dy = \int 3x^2\,dx$ 1: Seperated variable x y

$\frac{e^{2y}}{2} = \frac{\cancel{3}x^3}{\cancel{3}} + C$ 3: 1- Integrate Right, Integrate left, & +C

$\frac{1}{2}e^{2y} = x^3 + C$

$\frac{1}{2}e^{\cancel{2}(\frac{1}{\cancel{2}})} = (0)^3 + C$ 1: Using the initial condition

$\frac{1}{2}e = C$

$2\left(\frac{1}{2}e^{2y} = x^3 + \frac{1}{2}e\right)$

$\cancel{\ln e}^{2y} = \ln(2x^3 + e)$

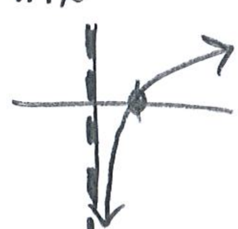
$\frac{1}{\cancel{2}}\cancel{2}y = \frac{1}{2}\ln(2x^3 + e)$

$\boxed{y = \frac{1}{2}\ln(2x^3 + e)}$ 1: Solving for y

b.) Find the domain and range of the function f found in part (a).

Parent Function

$y = \ln x$



domain: $(0, \infty)$
range: $(-\infty, \infty)$

$2x^3 + e > 0$ 1: set inside > 0

$2x^3 > -e$

$x^3 > -\frac{1}{2}e$

$x > \sqrt[3]{-\frac{1}{2}e}$

domain +1

range

$(-\infty, \infty)$ +1

Notes: Day1

AP Calculus

Wksht. Differential Equations: Additional Topics

Name________________________

Date__________________ Period__________

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point is (x, y) on the graph given by $\frac{dy}{dx} = \boxed{y^2(6-2x)}$.

a.) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

$$\frac{d}{dx}\left[\frac{dy}{dx} = \underbrace{y^2(6-2x)}_{\text{product rule}}\right]$$

$$\frac{d^2y}{dx^2} = y^2\frac{d}{dx}[6-2x] + (6-2x)\frac{d}{dx}[y^2]$$

$$= y^2(-2) + (6-2x)2y\boxed{\frac{dy}{dx}}$$

$$= y^2(-2) + (6-2x)(2y)y^2(6-2x)$$

$$-2y^2 + (6-2x)(2y)y^2(6-2x)$$

$$-2\left(\frac{1}{4}\right)^2 + \underbrace{\left[6-2(3)\right]}_{0}\left[2\left(\frac{1}{4}\right)\right]\left[\frac{1}{4}\right]^2\left[6-2(3)\right]$$

$$-2\left(\frac{1}{16}\right)$$

$$\boxed{-\frac{1}{8}}$$

b.) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6-2x)$ with the initial condition $f(3) = \frac{1}{4}$.

$x \quad y$

$$\frac{dy}{dx} = y^2(6-2x)$$

$$\frac{1}{y^2}dy = (6-2x)dx$$

$$\int y^{-2}\,dy = \int 6-2x\,dx$$

$$\frac{y^{-1}}{-1} = 6x - \frac{2x^2}{2} + C$$

$$-\frac{1}{y} = 6x - x^2 + C$$

$$-\frac{1}{\frac{1}{4}} = 6(3) - (3)^2 + C$$

$$-4 = 18 - 9 + C$$

$$-4 = 9 + C$$

$$-13 = C$$

$$-1\left(-\frac{1}{y} = 6x - x^2 - 13\right)$$

$$\frac{1}{y} = x^2 - 6x + 13 \qquad y = \frac{1}{x^2-6x+13}$$

Notes: Day 1

AP Calculus

Wksht. Differential Equations: Additional Topics

Name________________________

Date________________ Period____________

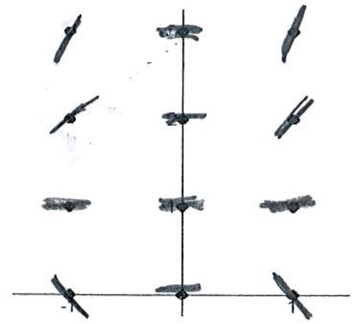
Consider the differential equation $\frac{dy}{dx} = x^2(y-1)$. Solve Equilibreuim first $\left(\frac{dy}{dx}=0\right)$

a.) On the axis provided, sketch a slope field for the given differential equation at the twelve points indicated.

$(-1,3) = (-1)^2(3-1) = 2 = (1,3)$

$(-1,2) = (-1)^2(2-1) = 1 = (1,2)$

$(-1,0) = (-1)^2(0-1) = -1 = (1,0)$



$\frac{dy}{dx} = x^2(y-1)$

$x^2=0 \quad y-1=0$

$x=0 \quad y=1$

b) While the slope field in part (a.) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are positive.

$\frac{dy}{dx} = x^2(y-1)$

$y-1>0$

$y>1$ then $\frac{dy}{dx}$ is positive

& $x \neq 0$

c.) Find the particular solution $y=f(x)$ to the given differential equation with the initial condition $f(0)=3$.

$\frac{dy}{dx} = x^2(y-1)$

$\int \frac{1}{y-1}\,dy = \int x^2\,dx$

$u=y-1$

$du=dy$

$\int \frac{1}{u}\,du = \frac{x^3}{3}+C$

$\ln|y-1| = \frac{1}{3}x^3+C$

$\ln 2 = C$

$e^{\ln|y-1|} = e^{\frac{1}{3}x^3+\ln(2)}$

$|y-1| = e^{\frac{1}{3}x^3}e^{\ln(2)}$

$|y-1| = 2e^{\frac{1}{3}x^3}$

$y = 1 \pm 2e^{\frac{1}{3}x^3}$

$3 = 1 \pm 2e^0$

Remember

$x^a \cdot x^b = x^{a+b}$

$$\boxed{y = 1+2e^{\frac{1}{3}x^3}}$$

CW: Day 1

AP Calculus
Wksht. Differential Equations: Additional Topics

Name________________
Date________________ Period________________

Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

$2x = 0 \quad x = 0$ (Equilibrium)

a.) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.

$(-1,2) = \frac{-2(-1)}{2} = 1$

$(-1,1) = \frac{-2(-1)}{1} = 2$

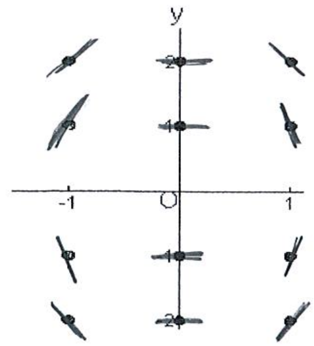
$(-1,-1) = \frac{-2(-1)}{-1} = -2$

$(-1,-2) = \frac{-2(-1)}{-2} = -1$

$(1,2) = \frac{-2(1)}{2} = -1$

$(1,1) = \frac{-2(1)}{1} = -2$

$(1,-1) = \frac{-2(1)}{-1} = 2$



$(1,-2) = \frac{-2(1)}{-2} = 1$

+2

b.) Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at (1,-1) and use it to approximate $f(1.1)$.

Tangent line

$\boxed{1}$ Point $(1,-1)$

$\boxed{2}$ Slope m at $(1,-1) = \frac{-2(1)}{-1} = 2$

$y + 1 = 2(x-1)$ +1

$y = 2(x-1) - 1$

$y(1.1) = 2(1.1-1) - 1$

$= 2(.1) - 1 = .2 - 1 = \boxed{-.8}$ +1

c.) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.

x y

$y\,dy = -2x\,dx$ +1

$\int y\,dy = \int -2x\,dx$

$\frac{y^2}{2} = \frac{-2x^2}{2}$ +1 $+C$ +1

$\frac{1}{2}y^2 = -x^2 + C$

$\frac{1}{2}(-1)^2 = -(1)^2 + C$ +1

$\frac{1}{2} = -1 + C$

$C = \frac{3}{2}$

$2\left(\frac{1}{2}y^2 = -x^2 + \frac{3}{2}\right)$

$y^2 = -2x^2 + 3$

$y = \pm\sqrt{-2x^2+3}$

$-1 = \pm\sqrt{-2(1)^2+3}$

$-1 = \pm\sqrt{1}$

$y = -\sqrt{-2x^2+3}$ +1

CW: Day 1

AP Calculus
Wksht. Differential Equations: Additional Topics

Name
Date Period

Equilibrium: $x=0$
$1+y=0 \quad y=-1$

Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$, where $x \neq 0$.

a.) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.

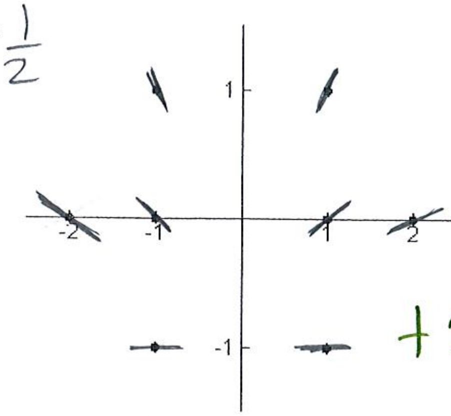
$(-2,0) = \frac{1+0}{-2} = \frac{1}{-2}$ $\quad (2,0) = \frac{1+0}{2} = \frac{1}{2}$

$(-1,1) = \frac{1+1}{-1} = -2$

$(-1,0) = \frac{1+0}{-1} = -1$

$(1,1) = \frac{1+1}{1} = 2$

$(1,0) = \frac{1+0}{1} = 1$



+2

b.) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

x y

$\frac{dy}{dx} = \frac{1+y}{x}$

$\frac{1}{1+y}\, dy = \frac{1}{x}\, dx$ +1

$\int \frac{1}{1+y}\, dy = \int \frac{1}{x}\, dx$

$u = 1+y$
$du = dy$

$\int \frac{1}{u}\, du = \int \frac{1}{x}\, dx$

+1 $\ln|1+y| = \ln|x|$ +1 $+C$ +1

$\ln|1+1| = \ln|-1| + C$ +1

$C = \ln(2)$

$e^{\ln|1+y|} = e^{\ln|x| + \ln(2)}$

$|1+y| = e^{\ln|x|} \cdot e^{\ln(2)}$

$|1+y| = 2|x|$

$1+y = \pm 2|x|$

$y = -1 \pm 2|x|$

$1 = -1 + 2|-1|$

$y = -1 + 2|x|$ +1

domain $= -\infty, \infty$ +1

CW: Day 1

AP Calculus

Wksht. Differential Equations: Additional Topics

Name________________________________

Date______________________ Period__________

Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

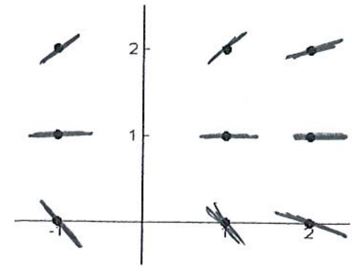
Equilibrium $y-1=0$
$y=1$

a.) On the axis provided, sketch a slope field for the given differential equation at the nine points indicated.

$(-1,2) = \frac{2-1}{(-1)^2} = 1$ $\quad (1,0) = \frac{0-1}{(1)^2} = -1$

$(-1,0) = \frac{0-1}{(-1)^2} = -1$ $\quad (2,2) = \frac{2-1}{4} = \frac{1}{4}$

$(1,2) = \frac{2-1}{(1)^2} = 1$ $\quad (2,0) = \frac{0-1}{4} = -\frac{1}{4}$



+2

b.) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

x y

$\frac{dy}{dx} = \frac{y-1}{x^2}$

$\frac{1}{y-1}\,dy = \frac{1}{x^2}\,dx$ +1

$\int \frac{1}{y-1}\,dy = \int x^{-2}\,dx$

$u = y-1$

$du = dy$

$\int \frac{1}{u}\,du = \int x^{-2}\,dx$

+1 $\ln|y-1| = \frac{x^{-1}}{-1} + C$ +1 +1

$\ln|y-1| = \frac{-1}{x} + C$

$\ln|0-1| = \frac{-1}{2} + C$ +1

$\frac{1}{2} = C$

$\ln|y-1| = \frac{-1}{x} + \frac{1}{2}$

$e \qquad e$

$|y-1| = e^{-\frac{1}{x}+\frac{1}{2}}$

$y = 1 \pm e^{-\frac{1}{x}+\frac{1}{2}}$

$0 = 1 \pm e^{-\frac{1}{2}+\frac{1}{2}}$

$y = 1 - e^{-\frac{1}{x}+\frac{1}{2}}$ +1

c.) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x\to\infty} f(x)$.

$\lim_{x\to\infty} 1 - e^{-\frac{1}{x}+\frac{1}{2}} = 1 - e^{\frac{-1}{\infty}+\frac{1}{2}}$

$= 1 - e^{1/2}$ +1