Notes: Optimization

Day 2

Suppose a company makes a profit of $P(x) = \frac{1000}{x} - \frac{5000}{x^2} + 100$ makes and sells $x > 0$ items. How many items should it make to maximize profit?

Maximize Profit

$\frac{d}{dx}\left[P(x) = 1000x^{-1} - 5000x^{-2} + 100\right]$

$0 = -1000x^{-2} + 10{,}000x^{-3}$

$-\frac{1000x}{x \cdot x^2} + \frac{10{,}000}{x^3} = 0$

$\frac{-1000x + 10{,}000}{x^3} = 0$

$-1000x + 10{,}000 = 0$

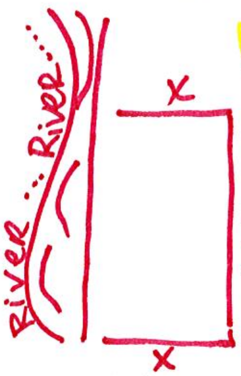
$1000x = 10{,}000$

$\boxed{x = 10}$

$+ \quad 10 \ \text{max} \quad -$

$P'(9) = +$ $\quad P'(11) = -$

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fencing along the river. What are the dimensions of the field that has the largest area?



Maximize Area

$A = \text{length} \cdot \text{width}$

$A = x \cdot y$

$A = x(2400 - 2x)$

$\frac{d}{dx}\left[A = 2400x - 2x^2\right]$

$P = 2400$

$2400 = 2x + y$

$2400 - 2x = y$

$0 = 2400 - 4x$

$4x = 2400$

$x = 600$

$+ \quad 600 \quad -$

$A'(599) = +$

$A'(601) = -$

dimensions $= x \cdot y \cdot x$

$\boxed{= 600 \cdot 1200 \cdot 600}$

$2400 - 2(600) = y$

$2400 - 1200 = y$

$1200 = y$

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1,4)$.

Minimize Distance

$y = \sqrt{2x}$ $\quad$ Points: $(1,4)$ & $(x, \sqrt{2x})$ $\quad \boxed{(2,2)}$

$\text{distance} = \sqrt{(x-1)^2 + (\sqrt{2x} - 4)^2}$

$\text{distance} = \sqrt{x^2 - \cancel{2x} + 1 + \cancel{2x} - 8\sqrt{2x} + 16}$

$\text{distance} = \sqrt{x^2 - 8(2x)^{1/2} + 17}$

$\frac{d}{dx}\left[\text{distance} = (x^2 - 8(2x)^{1/2} + 17)^{1/2}\right]$

$0 = \frac{1}{2}(x^2 - 8(2x)^{1/2} + 17)^{-1/2}\left[2x - 4(2x)^{-1/2}(2)\right]$

$0 = \frac{2x - 8(2x)^{-1/2}}{2\sqrt{x^2 - 8(2x)^{1/2} + 17}}$

$0 = \frac{\sqrt{2x}(2x)}{\sqrt{2x}} - \frac{8}{\sqrt{2x}} = \frac{(2x)^{3/2} - 8}{\sqrt{2x}}$

$(2x)^{3/2} - 8 = 0$

$((2x)^{3/2})^{2/3} = (8)^{2/3}$

$2x = 4$

$x = 2$

$- \quad 2 \ \text{min} \quad +$

$d'(1) \quad d'(3)$

A rectangular storage container with open top is to have a volume of $10\ m^3$. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materialsw for the cheapest such container.

$V = 10$ $\quad 10 = 2x \cdot x \cdot y = 2x^2y$

$\frac{10}{2x^2} = \frac{2x^2 y}{2x^2}$

$y = \frac{5}{x^2}$

Minimize Cost

$\text{Cost} = \text{Price}\left(\begin{smallmatrix}\text{Area}\\ \text{Base}\end{smallmatrix}\right) + \text{Price}\left(\begin{smallmatrix}\text{2 side}\\ \text{Area}\end{smallmatrix}\right) + \text{Price}\left(\begin{smallmatrix}\text{2 sides}\\ \text{Area}\end{smallmatrix}\right)$

$\text{Cost} = 10(2x \cdot x) + 6 \cdot 2(2x \cdot y) + 6 \cdot 2(xy)$

$\text{Cost} = 20x^2 + 24xy + 12xy$

$\text{Cost} = 20x^2 + 36xy$

$\text{Cost} = 20x^2 + 36x\left(\frac{5}{x^2}\right)$

$\frac{d}{dx}\left[\text{Cost} = 20x^2 + 180x^{-1}\right]$

$0 = 40x - 180x^{-2}$

$0 = \frac{x^2}{x^2} 40x - \frac{180}{x^2}$

$0 = \frac{40x^3 - 180}{x^2}$

$40x^3 - 180 = 0$

$40x^3 = 180$

$x^3 = 4.5$

$x = 1.651$

$- \quad 1.651 \ \text{min} \quad +$

$c'(1) = - \quad c'(2) = +$

$\text{Cost} = 20x^2 + \frac{180}{x}$

$\text{Cost} = 20(1.651)^2 + \frac{180}{1.651}$

$\boxed{\text{Cost} = \$163.54}$

A farmer has 400 feet of fencing to make three rectangular pens. What dimensions x and y will maximize the total area?

max $33\frac{1}{3}$

+ | −

$A'(33) = +$ $\quad A'(34) = -$

Area is Maximum

Area = length · width

$Area = y(3x)$

$Area = 3xy$

$Area = 3x(100 - 1.5x)$

$\frac{d}{dx}\left[Area = 300x - 4.5x^2\right]$

$P = 400$

$400 = 6x + 4y$

$400 - 6x = 4y$

$y = 100 - 1.5x$

$0 = 300 - 9x$

$9x = 300$

$x = 33\frac{1}{3}$

$y = 100 - 1.5(33\frac{1}{3})$

$x = 33\frac{1}{3}$

$y = 50$

What dimensions minimize the surface area of a can with volume 16π cubic inches?

SA = Minimum

$SA = 2(\text{Area top/bottom}) + (\text{Area Rectangle})$

$SA = 2(\pi R^2) + 2\pi Rh$

$SA = 2\pi R^2 + 2\pi R\left(\frac{16}{R^2}\right)$

$\left[SA = 2\pi R^2 + 32\pi R^{-1}\right]\frac{d}{dR}$

$V = 16\pi$

$\frac{16\pi}{\pi R^2} = \frac{\cancel{\pi R^2} h}{\cancel{\pi R^2}}$

$h = \frac{16}{R^2}$

$0 = 4\pi R - 32\pi R^{-2}$

$0 = R^2\frac{4\pi R}{R^2} - \frac{32\pi}{R^2}$

$0 = 4\pi R^3 - 32\pi$

$\frac{\cancel{4\pi} R^3}{\cancel{4\pi}} = \frac{32\pi}{4\pi}$

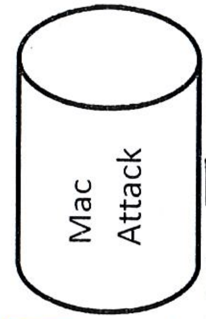
$R^3 = 8$

$R = 2$

− 2 +

min

$SA'(1) = -$ $\quad SA'(3) = +$



$h = \frac{16}{2^2}$

$h = \frac{16}{4}$

$h = 4$

$\boxed{r = 2 \quad h = 4}$

A painter has enough paint to cover 600 square feet of area. What is the largest square-bottom box that could be painted (including the top, bottom, & all sides)?

Maximum Volume

$V = l \cdot w \cdot h$

$V = x \cdot x \cdot y$

$V = x^2 y$

$y = \frac{150}{10} - \frac{1}{2}(10)$

$15 - 5 = 10$

$A = 600$

$A = 2(x \cdot x) + 4(x \cdot y)$

$600 = 2x^2 + 4xy$

$\frac{600 - 2x^2}{4x} = \frac{\cancel{4x}y}{\cancel{4x}}$

$y = \frac{150}{x} - \frac{1}{2}x$

$V = x^2 y$

$V = x^2\left(\frac{150}{x} - \frac{1}{2}x^2\right)$

$\frac{d}{dx}\left[V = 150x - \frac{1}{2}x^3\right]$

$0 = 150 - \frac{3}{2}x^2$

$\cancel{\frac{2}{3}}\cancel{\frac{3}{2}}x^2 = 150 \cdot \frac{2}{3}$

$x^2 = 100$

$x = 10$

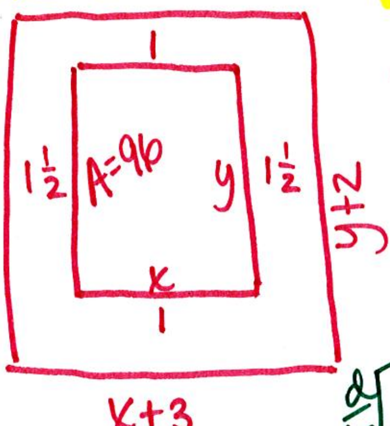
max 10

+ | −

$V'(9)$ $\quad V'(11)$

$\boxed{10 \cdot 10 \cdot 10}$

A printed page will have a total area of 96 square inches. The top and bottom margins will be 1 inch each, and the left and right margins will be $1\frac{1}{2}$ inches each. What overall dimensions for the page will maximize the area of the space inside the margins?



Area = Max

$Area = (x+3)(y+2)$

$Area = xy + 2x + 3y + 6$

$Area = x\left(\frac{96}{x}\right) + 2x + 3\left(\frac{96}{x}\right) + 6$

$Area = 96 + 2x + 288x^{-1} + 6$

$\frac{d}{dx}\left[Area = 104 + 2x + 288x^{-1}\right]$

$0 = 2 - 288x^{-2}$

$\frac{288}{x^2} = 2$ $\quad 2x^2 = 288$ $\quad x^2 = 144$ $\quad x = 12$

$x \cdot y = 96$

$y = \frac{96}{x}$

$y = \frac{96}{12} = 8$

max 12

$A'(11) = +$ $\quad A'(13) = -$

dimensions

$(x+3) \cdot (y+2)$

$\boxed{15 \cdot 10}$

An artist can sell 20 copies of a painting at $100 each, but for each additional copy she makes, the value of each painting will go down by a dollar. Thus, if 22 copies are made, each will sell for $98. How many copies should she make to maximize her sales?

Max Sales

Sales = (# copies)(price/copy)

$\text{Sales} = (20+x)(100-x)$

$\text{Sales} = 200 - 20x + 100x - x^2$

$\left[\text{Sales} = 200 + 80x - x^2\right]\frac{d}{dx}$

$0 = 80 - 2x$

$2x = 80$

$x = 40$

$\nearrow \overset{\text{max}}{40} \searrow$

$S'(39) = +$

$S'(41) = -$

$\#\text{Copies} = 20 + x$

$= 20 + 40$

$\boxed{= 60 \text{ Paintings}}$

A garden has 200 pounds of waterme3lons growing in it. Every day, the total amount of watermelon increases by 5 pounds. At the same time, the price per pound of watermelon goes down by 1 cent. If the current price is 90 cents per pound, how much longer should the watermelons grown in order to fetch the highest price possible.

Max Price

Price = (# Pounds)(cost/Pound)

$\text{Price} = (200 + 5x)(90 - 1x)$

$\text{Price} = 18{,}000 - 200x + 450x - 5x^2$

$\frac{d}{dx}\left[\text{Price} = 18{,}000 + 250x - 5x^2\right]$

$0 = 250 - 10x$

$10x = 250$

$x = 25$

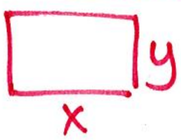
$\nearrow \overset{\text{max}}{25} \searrow$

$P'(24) = +$ $P'(26) = -$

$\boxed{x = 25 \text{ days}}$

A rectangular pen will be built using 100 feet of fencing. What dimensions will maximize the area?

Max Area = xy



$= x(50 - x)$

$\left[= 50x - x^2\right]\frac{d}{dx}$

$P = 100$

$100 = 2x + 2y$

$100 - 2x = 2y$

$50 - x = y$

$0 = 50 - 2x$

$2x = 50$

$x = 25$

$\nearrow \overset{\text{max}}{25} \searrow$

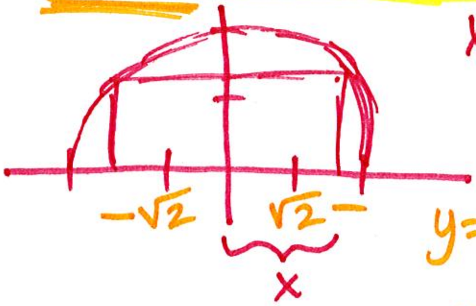
$A'(24) = +$ $A'(26) = -$

$\boxed{x = 25}$

$y = 50 - x$

$= 50 - 25 = \boxed{25}$

Find the area of the largest rectangle that can be inscribed in a semicircle of radius 2.



$x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \sqrt{4 - x^2}$

$y = \sqrt{4 - (\sqrt{2})^2}$

$= \sqrt{4 - 2}$

$= \sqrt{2}$

Area = length · width

$= (2\sqrt{2})(\sqrt{2})$

$= 2(2)$

$\text{Area} \boxed{= 4}$

Area Max

Area = length · width

$\left[\text{Area} = 2x\left(\sqrt{4 - x^2}\right)\right]\frac{d}{dx}$

$0 = 2x\frac{d}{dx}\left[(4 - x^2)^{1/2}\right] + \sqrt{4 - x^2}\,\frac{d}{dx}\left[2x\right]$

$0 = \cancel{2}x \cdot \frac{1}{\cancel{2}}(4 - x^2)^{-1/2}(-2x) + \sqrt{4 - x^2}\,(2)$

$0 = \frac{-2x^2}{\sqrt{4 - x^2}} + \frac{2\sqrt{4 - x^2}\sqrt{4 - x^2}}{\sqrt{4 - x^2}} = \frac{-2x^2 + 2(4 - x^2)}{\sqrt{4 - x^2}}$

$-2x^2 + 8 - 2x^2 = 0$

$-4x^2 + 8 = 0$

$4x^2 = 8$

$x^2 = 2$

$x = \sqrt{2}$

$\nearrow \overset{\text{max}}{\sqrt{2}} \searrow$

$A'(1) = +$

$A'(2) = -$

Suppose the profit of a company is when it makes $P(x) = 9x^2 + 40x - \frac{1}{3}x^3 + 1{,}000$ items a day. What level of production will maximize profits?

Maximize Profit

$$\frac{d}{dx}\left[\text{Profit} = 9x^2 + 40x - \tfrac{1}{3}x^3 + 1000\right]$$

$0 = 18x + 40 - x^2$

$x^2 - 18x - 40 = 0$

$(x+2)(x-20)$

$x = -2 \quad x = 20$

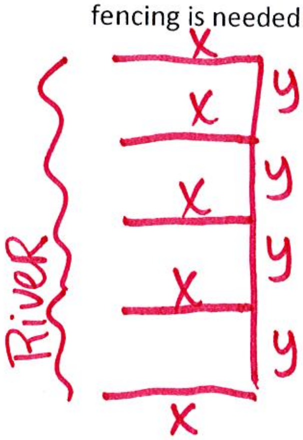
max

\+ 20 −

$P'(19) \quad P'(21)$

$\boxed{x = 20 \text{ items}}$

Four pens will be built along a river by using 100 feet of fencing. What dimensions will maximize the area? Assume no fencing is needed along the river.



Maximize Area

$A = \text{length} \cdot \text{width}$

$A = x(4y)$

$A = 4xy$

$A = 4x(25 - 1.25x)$

$$\frac{d}{dx}\left[A = 100x - 5x^2\right]$$

$0 = 100 - 10x \quad 10x = 100 \quad x = 10$

$P = 100$

$100 = 5x + 4y$

$100 - 5x = 4y$

$25 - 1.25x = y$

$y = 25 - 1.25(10)$

$y = 12.5$

max

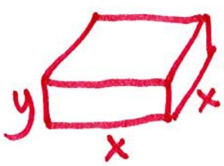
10

$A'(9) = + \quad A'(11) = -$

$\boxed{x = 10 \quad y = 12.5}$

A box with a square bottom will be built to contain 40,000 cubic feet of grain. The sides of the box cost 10 cents per square foot to build, the roof costs 1 dollar per square foot to build, and the bottom will cost 7 dollars per square foot to build. What dimension will minimize the building costs?

Minimize Cost



Mistake :(Redo :(

$\text{Cost} = .04\left(\begin{smallmatrix}\text{Area}\\ \text{4 sides}\end{smallmatrix}\right) \ldots (\begin{smallmatrix}\text{Area}\\ \text{top}\end{smallmatrix})$

$\text{Cost} = .04(4)(\ldots(x \cdot x)$

$\text{Cost} = .16\ldots$

$\text{Cost} = .16xy + \ldots$

$\text{Cost} = .16x\left(\dfrac{40{,}000}{x^2}\right) + 8x^2$

min

7.368

− +

$C'(7) = -$

$C'(8) = +$

$V = 40{,}000 \quad V = \text{length} \cdot \text{width} \cdot \text{height}$

$\dfrac{40{,}000}{x^2} = \dfrac{x \cdot x \cdot y}{x^2}$

$y = \dfrac{40{,}000}{x^2}$

$$\frac{d}{dx}\left[\text{Cost} = 6400x^{-1} + 8x^2\right]$$

$0 = -6400x^{-2} + 16x$

$0 = \dfrac{-6400}{x^2} + \dfrac{16x}{} \cdot x^2$

$-6400 + 16x^3 = 0$

$16x^3 = 6400$

$x^3 = 400$

$x = 7.368$

$\text{Cost} = \dfrac{6400}{x} + 8x^2$

$\text{Cost} = \dfrac{6400}{7.368} + 8(7.368)^2$

$\boxed{C = 1302.92}$

When 30 orange trees are planted on an acre, each will produce 500 oranges a year. For every additional orange tree planted, each tree will produce 10 fewer oranges. How many tress should be planted to maximize the yield?

Maximize Yield

Oranges = (# trees)(yield/tree)

Oranges = $(30 + x)(500 - 10x)$

Oranges = $15000 - 300x + 500x - 10x^2$

$$\frac{d}{dx}\left[\text{Oranges} = 15000 + 200x - 10x^2\right]$$

$0 = 200 - 20x$

$20x = 200$

$x = 10$

max

\+ 10 +

$O'(9) = + \quad O'(11) =$

trees = 30 + x

= 30 + 10

= $\boxed{40 \text{ trees}}$