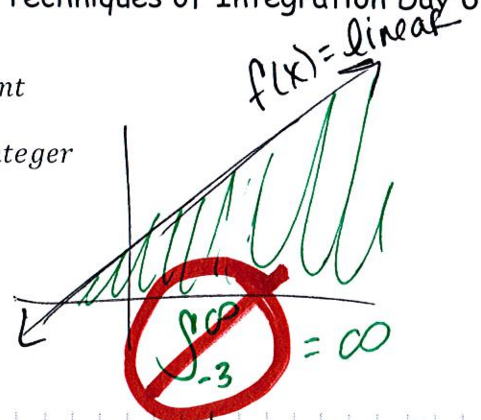
Notes: Improper Integrals

Remember: $\boxed{\lim_{x\to\infty}}\frac{c}{x^n} = 0$ where

End Behavior

$c = some\ constant$

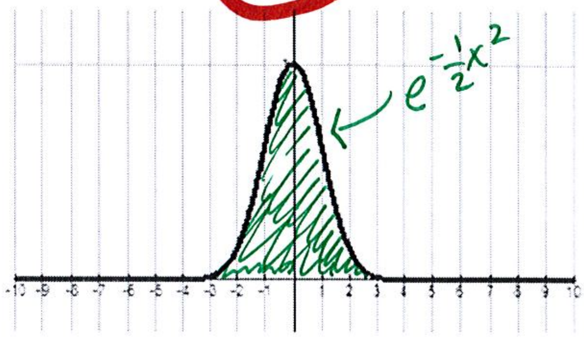
$n = some\ positive\ integer$



3 Types of Improper Integrals

1. $\int_{-\infty}^{b} f(x)dx$ 2. $\int_{a}^{\infty} f(x)dx$ 3. $\int_{-\infty}^{\infty} f(x)dx$

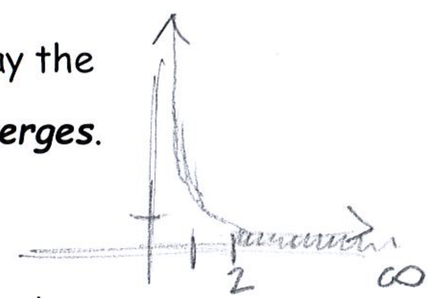
Example: $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx$ = Area



Improper Integrals: Although the region under the curve extends infinitely to the left and right, the total area is finite.

Converges: If the area approaches some number then the integral is said to converge.

$\int_{a}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{a}^{R} f(x)dx$ When the limit exists, we say the improper integral ***converges***.



Example One: Show that $\int_{2}^{\infty}\frac{dx}{x^2}$ converges and compute its value.

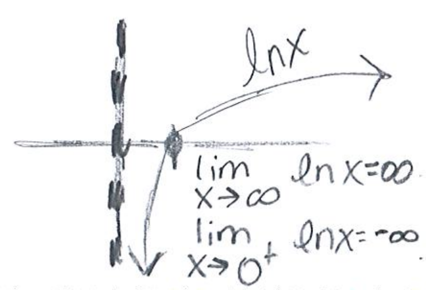
$\lim_{R\to\infty}\int_{2}^{R} x^{-2}dx = \lim_{R\to\infty}\frac{x^{-1}}{-1}\Big|_{2}^{R} = \lim_{R\to\infty}\frac{-1}{x}\Big|_{2}^{R}\ \lim_{R\to\infty}\underbrace{-\frac{1}{R}}_{0}-\left(-\frac{1}{2}\right) = \frac{1}{2}$

converges

Example Two: Determine if $\int_{-\infty}^{-1}\frac{dx}{x}$ converges.

$\lim_{R\to-\infty}\int_{R}^{-1}\frac{1}{x}dx = \lim_{R\to-\infty}\ln|x|\Big]_{R}^{-1} = \lim_{R\to-\infty}\underbrace{\ln|-1|}_{0} - \ln|R| = -\infty$

does not converge.



Example Three: Determine if $\int_0^{\infty} xe^{-x}\,dx$ converges. If so compute it's value.

$$\lim_{R\to\infty}\int_0^R xe^{-x}\,dx \quad \frac{u=x}{du=dx} \quad \frac{v=-e^{-x}}{dv=e^{-x}dx} \quad \lim_{R\to\infty} -xe^{-x}+\int_0^R e^{-x}\,dx$$

$$\lim_{R\to\infty} -xe^{-x}-e^{-x}\Big|_0^R = \lim_{R\to\infty}\frac{-x}{e^x}-\frac{1}{e^x}\Big|_0^R = \lim_{R\to\infty}\left(\frac{-R}{e^R}-\frac{1}{e^R}\right)-\left(\frac{-0}{e^0}-\frac{1}{e^0}\right)=\boxed{1}$$

0 0 0

$\frac{\infty}{\infty}$ = Indeterminant So LHopitals Rule

$$\lim_{R\to\infty}\frac{[-R]\frac{d}{dR}}{[e^R]\frac{d}{dR}}$$

$$\lim_{R\to\infty}\frac{-1}{e^R}=0$$

Memorize (if you want)

$$\int_1^{\infty}\frac{1}{x^p}\,dx=\begin{cases}\frac{1}{p-1} & if\ p>1\\ \infty & if\ p\le 1\end{cases}$$

Infinite Discontinuities at endpoints:

If $f(x)$ is continuous on $[a,b)$ but discontinuous at $x=b$ we define

$$\int_a^b f(x)dx=\lim_{x\to b^-}\int_a^R f(x)dx$$

If $f(x)$ is continuous on $(a,b]$ but discontinuous at $x=a$ we define

$$\int_a^b f(x)dx=\lim_{x\to a^+}\int_R^b f(x)dx$$

If, in either case, the limit does not exist, the integral is said to ***Diverge***.

$[0,1)$ X=1 is not in the domain of $f(x)=\frac{1}{\sqrt{1-x^2}}$

Example Four: $\int_0^1\frac{1}{\sqrt{1-x^2}}\,dx$

$$\lim_{R\to1^-}\int_0^R\frac{1}{\sqrt{1-x^2}}\,dx=\lim_{R\to1^-}\sin^{-1}(x)\Big|_0^R=\lim_{R\to1^-}\sin^{-1}(R)-\sin^{-1}(0)$$

$$\sin^{-1}(1)-\sin^{-1}(0)$$

$$\frac{\pi}{2}-0$$

$$\boxed{\frac{\pi}{2}}$$:)

$(0,1]$

Example Five: $\int_0^1\frac{1}{\sqrt{x}}\,dx$

$$\lim_{R\to0^+}\int_R^1 x^{-\frac{1}{2}}\,dx=\lim_{R\to0^+}2x^{\frac{1}{2}}\Big|_R^1=\lim_{R\to0^+}2\sqrt{1}-2\sqrt{R}=\boxed{2}$$

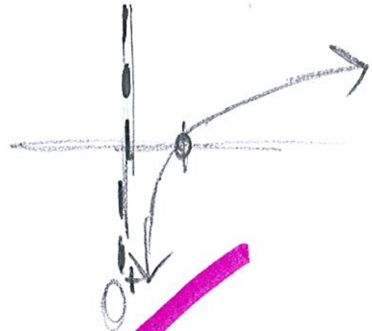
0

(0,1]

Example Six: $\int_0^1 \frac{1}{x}dx \; \lim_{R\to 0^+}\int_R^1 \frac{1}{x}dx = \lim_{R\to 0^+} \ln|x|\Big]_R^1 = \lim_{R\to 0^+} \ln|1| - \ln|R|$

0 $\quad --\infty$

$\boxed{\infty}$



diverges

Memorize (if you want)

$$\int_0^1 \frac{1}{x^p}dx = \begin{cases} & if\ p < 1 \\ & if\ p \geq 1 \end{cases}$$

Comparison Test for Improper Integrals: Assume that $f(x) \geq g(x) \geq 0$ for $x \geq a$

If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ also converges.

If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ also diverges.

Save until Unit 3

Example Seven: Show that $\int_1^\infty \frac{1}{\sqrt{x^3+1}}dx$ converges.

Example Eight: Determine if $\int_0^\infty \frac{1}{\sqrt{x^2+1}}dx$ converges or diverges.