Additional Integration 10

When is an Integration Problem Improper?

1. If you have infinity for at least one of your limits:

$\Rightarrow \int_{-\infty}^{\#} f(x)dx = \lim_{R\to-\infty} \int_R^{\#} f(x)dx \Rightarrow \int_{\#}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{\#}^{R} f(x)dx$

$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to-\infty} \int_R^{0} f(x)dx + \lim_{R\to\infty} \int_0^{R} f(x)dx$

2. Discontinuous at endpoint or on interval:

let b not be in the domain of f(x).

$\Rightarrow \int_a^b f(x)dx = \lim_{R\to b^-} \int_a^R f(x)dx \Rightarrow \int_b^c f(x)dx = \lim_{R\to b^+} \int_R^c f(x)dx$

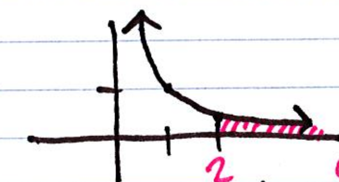
$\Rightarrow \int_a^c f(x)dx = \lim_{R\to b^-} \int_a^R f(x)dx + \lim_{R\to b^+} \int_R^c f(x)dx$

Additional Integration 11

When does an improper integral converge? diverge?

Converges: If the Integral has a numerical answer.

Ex $\int_2^{\infty} \frac{1}{x^2} dx = \frac{1}{2}$



diverges: If the Integral has answer ∞ or $-\infty$

Ex $\int_2^{\infty} x\, dx = \infty$

Additional Integration 12

$$\int_1^{\infty} \frac{1}{x^p} = \begin{cases} \_\_ & \text{if } p>1 \\ \_\_ & \text{if } p\le 1 \end{cases}$$

$$\int_1^{\infty} \frac{1}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p\le 1 \end{cases}$$

Ex $\int_1^{\infty} \frac{1}{x^3} dx \quad p=3 \quad = \frac{1}{3-1} = \boxed{\frac{1}{2}}$

Ex $\int_1^{\infty} \frac{1}{x^{1/3}} \quad p=\frac{1}{3} \quad = \boxed{\infty}$

Additional Integration 13

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{—} & p<1 \\ \text{—} & p\geq 1 \end{cases}$$

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p} \text{ if } p<1$$

$$\infty \text{ if } p\geq 1$$

Ex $\int_0^1 \frac{1}{x^3}\,dx \quad p=3 \quad$ so $= \boxed{\infty}$

Ex $\int_0^1 \frac{1}{x^{1/3}} \quad p=\frac{1}{3} \quad$ so $\frac{1}{1-1/3} = \frac{1}{\frac{2}{3}} = \boxed{\frac{3}{2}}$

Additional Integration 14

$$\int_1^\infty \frac{1}{x^{19/20}}\,dx$$

$$\int_1^\infty \frac{1}{x^{19/20}}\,dx = \lim_{R\to\infty} \int_1^R x^{-19/20}\,dx$$

$$= \lim_{R\to\infty} \frac{20}{1}\cdot x^{1/20}\Big|_1^R = \lim_{R\to\infty} 20R^{1/20} - 20(1)^{1/20}$$

∞ - some #

$= \boxed{\infty}$

$\therefore$ diverges

OR $\int_1^\infty \frac{1}{x^{19/20}} \quad p=\frac{19}{20} \quad \therefore p\leq 1$ so $\boxed{\infty}$

Additional Integration 15

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$$

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx$$

$$= \lim_{R\to-\infty} -\frac{1}{2}\int_R^0 -2xe^{-x^2}\,dx + \lim_{R\to\infty} -\frac{1}{2}\int_0^R -2xe^{-x^2}\,dx$$

$u=-x^2$
$du=-2x\,dx$

$$= \lim_{R\to-\infty} -\frac{1}{2}\int_R^0 e^u\,du + \lim_{R\to\infty} -\frac{1}{2}\int_0^R e^u\,du$$

$$= \lim_{R\to-\infty} -\frac{1}{2}e^u\Big|_R^0 + \lim_{R\to\infty} -\frac{1}{2}e^u\Big|_0^R$$

$$= \lim_{R\to-\infty} -\frac{1}{2}e^{-x^2}\Big|_R^0 + \lim_{R\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^R$$

$$\left(-\frac{1}{2e^0} + \frac{1}{2e^{(-\infty)^2}}\right) + \left(-\frac{1}{2e^{(\infty)^2}} + \frac{1}{2e^0}\right) = -\frac{1}{2} + \frac{1}{2} = 0$$

Additional Integration 16

$$\int_0^1 x \ln x \, dx$$

$$\int_{\boxed{0}}^1 x \ln x \, dx$$

0 is not in the domain of $\ln x$ so domain $(0, 1]$ so 0^+

Integration by Parts & u must be $\ln x$.

$$\lim_{R\to 0^+} \int_R^1 x \ln x \, dx \qquad \underline{u = \ln x} \quad v = \tfrac{1}{2}x^2$$
$$du = \tfrac{1}{x}dx \quad \underline{dv = x\,dx}$$

$$\lim_{R\to 0^+} \tfrac{1}{2}x^2 \ln x - \int_R^1 \tfrac{1}{2}x^2 \cdot \tfrac{1}{x}\,dx$$

$$\lim_{R\to 0^+} \tfrac{1}{2}x^2 \ln x - \int_R^1 \tfrac{1}{2}x \, dx$$

$$\lim_{R\to 0^+} \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\tfrac{x^2}{2}\Big|_R^1$$

$$\lim_{R\to 0^+} \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2\Big|_R^1$$

$$\lim_{R\to 0^+} \tfrac{1}{2}(1)^2\underbrace{\ln 1}_{0} - \tfrac{1}{4}(1)^2 - \underbrace{\tfrac{1}{2}R^2 \cdot \ln R}_{0\cdot(-\infty)\,\to\,0} + \underbrace{\tfrac{1}{4}R^2}_{0}$$

$$= -\frac{1}{4}$$

Additional Integration 17

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

1 is not in the domain of $1/\sqrt[3]{x-1}$ so break up $\int_0^1 + \int_1^9$

$$\lim_{R\to 1^-} \int_0^R \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{R\to 1^+} \int_R^9 \frac{1}{\sqrt[3]{x-1}}\,dx \qquad u = x-1 \quad du = dx$$

$$\lim_{R\to 1^-} \int_0^R u^{-1/3}\,du + \lim_{R\to 1^+} \int_R^9 u^{-1/3}\,du$$

$$\lim_{R\to 1^-} \tfrac{3}{2}u^{2/3}\Big|_0^R + \lim_{R\to 1^+} \tfrac{3}{2}u^{2/3}\Big|_R^9$$

$$\lim_{R\to 1^-} \tfrac{3}{2}(x-1)^{2/3}\Big|_0^R + \lim_{R\to 1^+} \tfrac{3}{2}(x-1)^{2/3}\Big|_R^9$$

$$\lim_{R\to 1^-} \underbrace{\tfrac{3}{2}(R-1)^{2/3}}_{0} - \tfrac{3}{2}(\sqrt[3]{0-1})^2 + \lim_{R\to 1^+} \tfrac{3}{2}(\sqrt[3]{9-1})^2 - \underbrace{\tfrac{3}{2}(R-1)^{2/3}}_{0}$$

$$-\tfrac{3}{2}(-1)^2 + \tfrac{3}{2}(2)^2$$
$$-\tfrac{3}{2} + \tfrac{12}{2} = \boxed{\tfrac{9}{2}}$$