Taylor Error: Actual and LaGrange

Actual Error
This is the real amount of error, not the error bound (worst case scenario). It is the difference between the actual $f(x)$ and the polynomial.

Steps:
1. Substitute x-value into $f(x)$ to get a value.
2. Substitute x-value into the polynomial and get another value.
3. The difference between the two is the error.

Example:

Approximate $f(.1)$ using a 2^{nd} degree Taylor polynomial centered at $a = 0$

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots$$

$$f(0.1) = \frac{1}{1-.1} = 1.111111\ldots$$

$$P_2(.1) = 1 + (.1) + (.1)^2 = 1.11$$

Error = $f(.1) - P_2(.1)$
= 1.111111… – 1.11
= **.001111….**

La Grange
This method uses a special form of the Taylor formula to find the **error bound** of a polynomial approximation of a Taylor series.

The LaGrange Formula:

$$\text{Error bound} = \frac{f^{n+1}(z)(x-a)^{n+1}}{(n+1)!}$$

The variable z is a number between x and a, but to find the error bound, z ends up being equal to one of the two. To determine whether the z value will be the same as x or a, you must substitute each number into $f^{n+1}(z)$ to see which gives the greatest number.

For example:

Approximate $f(.1)$ using a 2nd degree Taylor polynomial centered at $a = 0$

If you are trying to find the error of a 2nd degree Taylor polynomial approximation of $f(x) = \frac{1}{1-x}$, you must first find the 3rd derivative, because the formula uses $f^{n+1}(z)$, not $f^{n}(z)$.

$f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$, and $f'''(x) = \frac{6}{(1-x)^4}$

Also, for this function, $x = .1$ and $a = 0$. Substitute these two values into the 3rd derivative.

$$f^{3}(0) = \frac{6}{(1-0)^4} = 6$$

$$f^{3}(.1) = \frac{6}{(1-.1)^4} = \mathbf{9.145} \quad \leftarrow \textbf{this is bigger!}$$

Next, substitute in 9.145 for $f^{n+1}(z)$ in the La Grange formula:

$$\text{Error bound} = \frac{9.145(.1-0)^3}{3!} = \mathbf{.00152}$$

Exception!

$\Downarrow$

When $f(x) = \sin(x)$ or $\cos(x)$, the value for $f^{n+1}(z)$ will always be equal to 1, because that is the greatest value of any sine or cosine function.

Examples for La Grange error bound:

a. Find the upper bound for the error for the 5th degree polynomial approximation of *e*.

e is equal to e^1, whose series can be determined from the McLaurin series of e^x.

$$e^1 = \frac{(1)^0}{0!} + \frac{(1)^1}{1!} + \frac{(1)^2}{2!} + \frac{(1)^3}{3!} + \frac{(1)^4}{4!} + \frac{(1)^5}{5!}$$

The La Grange formula is,

$$\frac{f^{(6)}(z)(x)^6}{6!}$$

All derivatives of e^x are e^x ,so $f^6(z) = e^z$

To find *z*, substitute in the values for *a* and *x* into e^z.

$a = 0$, $e^0 = 1$
$x = 1$, $e^1 = e$

$e > 1$, so $z = 1$

$$\frac{e^1(1)^6}{6!} = \frac{e}{6!} = \frac{e}{720} = .00377$$

The actual error for this 5th degree polynomial falls somewhere between the real value of e^1 and e^1 + .00377.

$e^1 = 2.71828$

$e^1 \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.71666$

The error is 2.71828 – 2.71666, which equals 0.00162. This number is less than the upper bound for the error, 0.00377, which shows how the La Grange formula works.

b. What degree Taylor polynomial for ln(1.2) might have an error less than 0.001? (In other words, the upper bound for the error would be 0.001)

First, start off with the La Grange formula, whose value must be less than 0.001:

$$\frac{f^{n+1}(z)(x-a)^{n+1}}{(n+1)!} < 0.001$$

For the function ln(x), $a = 1$ and in this case, $x = 1.2$

The derivatives of ln(x) are as follows:

$$f(x) = \ln(x), f'(x) = \frac{1}{x}, f''(x) = \frac{-1}{x^2}, f'''(x) = \frac{2}{x^3}, f''''(x) = \frac{-6}{x^4}$$

Since you don't know the value of n, a general formula for the $n+1$ derivative must be used. The formula for the nth derivative can be obtained from above and is as follows:

$$f^n(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$$

To find $f^{n+1}(x)$, simply substitute $n+1$ for n into that equation:

$$f^{n+1}(x) = \frac{(-1)^{n+2}(n)!}{x^{n+1}}$$

This is what you will put into the La Grange formula for $f^{n+1}(z)$, changing x to z.

Still, you must find the value for z. It will be equal to either a or x. When substituting the two values into the above list of derivative for ln(x), you find that 1 always produces the greater value, so z = 1.

Now, the error bound formula looks something like this:

$$\frac{\frac{(-1)^{n+2}(n)!}{z^{n+1}}(x-a)^{n+1}}{(n+1)!} = \frac{\frac{(n)!}{z^{n+1}}(1.2-1)^{n+1}}{(n+1)!} = \frac{n!}{(n+1)!} \bullet \frac{(.2)^{n+1}}{z^{n+1}} = \frac{1}{n+1}\left(\frac{.2}{z}\right)^{n+1} < 0.001$$

$$\frac{1}{n+1}(.2)^{n+1} < 0.001$$

Next you must simply use the concept of trial and error. Choose values for n and keep substituting them in to the inequality. When the term on the left ends up being greater than 0.001, you know that you have crossed the line and your value for n will be the previous number (before the value exceeded 0.001).

$$\frac{1}{3+1}(.2)^{3+1} = .0004 < .001$$

$$\frac{1}{2+1}(.2)^{2+1} = .00267 > .001$$

The value for n is 3.