AP Calculus-BC
Review for Test: Additional Integration Techniques

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Integrate.

1. $\int \frac{3x-1}{x^2+3x-10}\,dx$ $\frac{3x-1}{(x+5)(x-2)} = \frac{A}{(x+5)} + \frac{B}{(x-2)}$

$3x-1 = A(x-2)+B(x+5)$

$\boxed{x=2}$
$3(2)-1 = B(7)$
$5 = 7B$
$B = 5/7$

$\boxed{x=-5}$
$3(-5)-1 = A(-7)$
$-16 = -7A$
$A = \frac{16}{7}$

$\int \frac{16/7}{x+5} + \frac{5/7}{x-2}$

$\boxed{\frac{16}{7}\ln|x+5| + \frac{5}{7}\ln|x-2| + C}$ ✓

2. $\int \cos^7 3x\,dx$ $u = \sin(3x)$ $du = 3\cos 3x\,dx$

$\int \cos(3x)\cos^2(3x)\cos^2(3x)\cos^2(3x)\,dx$

$\frac{1}{3}\int 3\cos(3x)[1-\sin^2(3x)][1-\sin^2(3x)][1-\sin^2(3x)]$

$\frac{1}{3}\int (1-u^2)(1-u^2)(1-u^2)\,du$

$\frac{1}{3}\int (1-u^2)(1-2u^2+u^4)\,du$

$\frac{1}{3}\int 1-2u^2+u^4-u^2+2u^4-u^6\,du$

$\frac{1}{3}\int 1-3u^2+3u^4-u^6\,du$

$\frac{1}{3}\left[u - \frac{\cancel{3}u^3}{\cancel{3}} + \frac{3u^5}{5} - \frac{u^7}{7} + C\right.$

$\frac{1}{3}u - \frac{1}{3}u^3 + \frac{1}{5}u^5 - \frac{1}{21}u^7 + C$

$\boxed{\frac{1}{3}\sin(3x) - \frac{1}{3}\sin^3(3x) + \frac{1}{5}\sin^5(3x) - \frac{1}{21}\sin^7(3x) + C}$

✓

3. $\int x^2 \ln x\,dx$ $u = \ln x$ $v = \frac{1}{3}x^3$ $du = \frac{1}{x}dx$ $dv = x^2\,dx$

$\frac{1}{3}x^3\ln x - \int \frac{1}{3}x^{\cancel{3}\,2}\left(\frac{1}{\cancel{x}}\right)dx$

$\frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2\,dx$

$\frac{1}{3}x^3\ln x - \frac{1}{3}\frac{x^3}{3} + C$

$\boxed{\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C}$ ✓

4. $\int x^2 e^x\,dx$ $u = x^2$ $v = e^x$ $du = 2x\,dx$ $dv = e^x\,dx$

$x^2e^x - \int 2xe^x\,dx$

$x^2e^x - 2\int xe^x\,dx$ $u = x$ $v = e^x$ $du = dx$ $dv = e^x\,dx$

$x^2e^x - 2\left[xe^x - \int e^x\,dx\right.$

$\boxed{x^2e^x - 2xe^x + 2e^x + C}$ ✓

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5. $\int \sin^2 x \cos^2 x \, dx$

$\int \frac{1}{2}[1-\cos(2x)]\frac{1}{2}[1+\cos(2x)]\,dx$

$\frac{1}{4}\int [1-\cos(2x)][1+\cos(2x)]$

$\frac{1}{4}\int 1-\cos^2(2x)\,dx$

$\frac{1}{4}\int 1-\frac{1}{2}[1+\cos(4x)]$

$\frac{1}{4}\int 1-\frac{1}{2}-\frac{1}{2}\cos(4x)\,dx$

$\frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos(4x)\,dx$

$\frac{1}{4}\left[\frac{1}{2}x-\frac{1}{2}\frac{\sin(4x)}{4}\right)+C$

$\boxed{\frac{1}{8}x-\frac{1}{32}\sin(4x)+C}$ ✓

6. $\int \frac{2x^2+17x+16}{x^2+8x+7}\,dx$

$\frac{x+2}{(x+7)(x+1)} = \frac{A}{x+7}+\frac{B}{x+1}$

$$\begin{array}{r} 2 \\ x^2+8x+7\overline{)\,2x^2+17x+16} \\ -2x^2\mp 16x\mp 14 \\ \hline x+2 \end{array}$$

$\int 2+\frac{x+2}{(x+7)(x+1)}$

$x+2 = A(x+1)+B(x+7)$

$\boxed{x=-1}$ $1=6B$, $B=1/6$

$\boxed{x=-7}$ $-5=-6A$, $A=5/6$

$\int 2+\int\frac{x+2}{(x+7)(x+1)}$

$2x+\int\frac{5/6}{x+7}+\frac{1/6}{x+1}$

$\boxed{2x+\frac{5}{6}\ln|x+7|+\frac{1}{6}\ln|x+1|+C}$ ✓

7. $\frac{1}{2}\int 2x\sec(x^2+4)\,dx$ $u=x^2+4$, $du=2x\,dx$

$\frac{1}{2}\int \sec u\,du$

$\frac{1}{2}\ln|\sec u+\tan u|+C$

$\boxed{\frac{1}{2}\ln|\sec(x^2+4)+\tan(x^2+4)|+C}$ ✓

8. $\int \frac{x+2}{x^2+4}\,dx$

$\frac{1}{2}\int\frac{2x}{x^2+4}+\int\frac{2}{x^2+4}\,dx$

$u=x^2+4$, $du=2x\,dx$ $\quad +2\int\frac{1}{4\left(\frac{x^2}{4}+1\right)}\,dx$

$\frac{1}{2}\int\frac{1}{u}\,du+2\left(\frac{1}{4}\right)\cdot 2\int\frac{\frac{1}{2}\cdot 1}{\left(\frac{x}{2}\right)^2+1}\,dx$ $\quad u=\frac{1}{2}x$, $du=\frac{1}{2}dx$

$\frac{1}{2}\ln|u| + \int\frac{1}{u^2+1}\,du$

$\frac{1}{2}\ln|x^2+4|+\tan^{-1}u+C$

$\boxed{\frac{1}{2}\ln|x^2+4|+\tan^{-1}\left(\frac{1}{2}x\right)+C}$ ✓

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$u = \cos x \quad du = -\sin x\,dx$

9. $\int e^x \sin x\,dx$ $\quad \underline{u = e^x}$ $\quad v = -\cos x$
$du = e^x dx \quad \underline{dv = \sin x\,dx}$

$-e^x \cos x - \int -e^x \cos x\,dx$ $\quad \underline{u = e^x} \quad v = \sin x$
$-e^x \cos x + \int e^x \cos x\,dx$ $\quad du = e^x dx \quad \underline{dv = \cos x\,dx}$

$-e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$

$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$
$+\int e^x \sin x\,dx \qquad \cancel{+\int e^x \sin x\,dx}$

$\frac{1}{2}\left(2\int e^x \sin x\,dx\right) = \left(-e^x \cos x + e^x \sin x + C\right)\frac{1}{2}$

$$\boxed{= -\frac{1}{2}e^x \cos x + \frac{1}{2}e^x \sin x + C}$$ ✓

10. $\int \sin^5 x\,dx$

$\int \sin x \cdot \sin^2 x \cdot \sin^2 x\,dx$

$-\int -\sin x(1-\cos^2 x)(1-\cos^2 x)\,dx$

$-\int (1-u^2)(1-u^2)\,du$

$-\int 1 - 2u^2 + u^4\,du$

$-\left[u - \frac{2}{3}u^3 + \frac{u^5}{5} + C\right.$

$-u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C$

$$\boxed{-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C}$$ ✓

11. $\int \tan^2 2x\,dx$

Pythagorean Identities

$\tan^2 x + 1 = \sec^2 x$
$\tan^2 x = \sec^2 x - 1$

$\int \sec^2(2x) - 1\,dx$

$\frac{\tan(2x)}{2} - x + C$

$$\boxed{\frac{1}{2}\tan(2x) - x + C}$$ ✓

12. $\int \frac{dx}{x^2 + 2x}$

$\int \frac{1}{x(x+2)} \qquad \frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

$\int \frac{\frac{1}{2}}{x} + \frac{-\frac{1}{2}}{x+2}\,dx \qquad 1 = A(x+2) + B(x)$

$\boxed{x = -2} \qquad \boxed{x = 0}$
$1 = B(-2) \qquad 1 = A(2)$
$B = -\frac{1}{2} \qquad A = \frac{1}{2}$

$$\boxed{\frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C}$$ ✓

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13. $\int e^x \cos x\,dx$ $\quad u=e^x \quad v=\sin x$
$du=e^x \quad dv=\cos x\,dx$

$e^x\sin x-\int e^x\sin x\,dx \quad u=e^x \quad v=-\cos x$
$du=e^x \quad dv=\sin x\,dx$

$e^x\sin x-[-e^x\cos x-\int -e^x\cos x$

$e^x\sin x+e^x\cos x-\int e^x\cos x$

$\int e^x\cos x\,dx=e^x\sin x+e^x\cos x-\int e^x\cos x\,dx$
$+\int e^x\cos x\,dx \qquad +\int e^x\cos x\,dx$

$\frac{1}{2}\left(2\int e^x\cos x\,dx\right)=\left(e^x\sin x+e^x\cos x+C\right)\frac{1}{2}$

$\boxed{=\frac{1}{2}e^x\sin x+\frac{1}{2}e^x\cos x+C}$ ✓

14. $\int_0^{\frac{\pi}{2}} x^3\cos 2x\,dx$

u	dv
$+x^3$	$\cos(2x)$
$-3x^2$	$\frac{\sin(2x)}{2}$
$+6x$	$\frac{-\cos(2x)}{4}$
-6	$\frac{-\sin(2x)}{8}$
$+0$	$\frac{\cos(2x)}{16}$

$\frac{1}{2}x^3\sin(2x)+\frac{3}{4}x^2\cos(2x)-\frac{3}{4}x\sin(2x)-\frac{3}{8}\cos(2x)\Big|_0^{\frac{\pi}{2}}$

$\cancel{\frac{1}{2}\left(\frac{\pi}{2}\right)^3\sin(\pi)}+\frac{3}{4}\left(\frac{\pi}{2}\right)^2\cos\pi-\cancel{\frac{3}{4}\left(\frac{\pi}{2}\right)\sin\pi}-\cancel{\frac{3}{8}\cos\pi}$
$-\cancel{\frac{1}{2}(0)^3}\quad-\cancel{\frac{3}{4}(0)^2}\quad+\cancel{\frac{3}{4}(0)}\quad+\cancel{\frac{3}{8}\cos(0)}$

$-\frac{3\pi^2}{4\cdot4}=\frac{-3\pi^2}{16}$

15. $\int \tan^5 x\sec^4 x\,dx$ $\quad u=\tan x$
$du=\sec^2x\,dx$

$\int \tan^5x\cdot\sec^2x\cdot\sec^2x\,dx$

$\int \tan^5x(\tan^2+1)\cdot\sec^2x\,dx$

$\int u^5(u^2+1)\,du$

$\int u^7+u^5\,du$

$\frac{u^8}{8}+\frac{u^6}{6}+C$

$\boxed{\frac{1}{8}\tan^8x+\frac{1}{6}\tan^6x+C}$ ✓

16. $\int_1^\infty xe^{-x}dx$ $\quad u=x \quad v=-e^{-x}$
$du=dx \quad dv=e^{-x}dx$

$\lim_{R\to\infty}\int_1^R xe^{-x}dx$

$\lim_{R\to\infty} -xe^{-x}-\int -e^{-x}dx$

$\lim_{R\to\infty} -xe^{-x}+\int e^{-x}dx$

$\lim_{R\to\infty} -xe^{-x}-e^{-x}\Big|_1^R$

$\lim_{R\to\infty}\left(-\frac{R}{e^R}-\cancel{\frac{1}{e^R}}\right)-\left(\frac{-1}{e^1}-\frac{1}{e^1}\right)$

$\lim_{R\to\infty}{}^{*} -\cancel{\frac{1}{e^R}}-0+\frac{2}{e^1}$

0

$0-0+\frac{2}{e}=\boxed{\frac{2}{e}}$ ✓

* l Hopitals Rule

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17. $\int \frac{3x\,dx}{\sqrt{3-x^2}}$

$u = 3 - x^2$

$du = -2x\,dx$

$-\frac{1}{2}3\int \frac{-2x}{\sqrt{3-x^2}}\,dx$

$-\frac{3}{2}\int \frac{1}{u^{1/2}}\,du$

$-\frac{3}{2}\int u^{-1/2}\,du$

$-\frac{3}{\cancel{2}}\cdot\frac{\cancel{2}}{1}u^{1/2}+C$

$\boxed{-3\sqrt{3-x^2}+C}$ ✓

18. $\int \csc x\,dx$

$= \boxed{\ln|\csc x - \cot x| + C}$ ✓

19. $\int \frac{2\,dx}{\sqrt{9-x^2}}$

$2\int \frac{1}{\sqrt{9-x^2}}\,dx$

$2\int \frac{1}{\sqrt{9(1-\frac{x^2}{9})}}\,dx$

$2\int \frac{1}{3\sqrt{1-(\frac{x}{3})^2}}\,dx$

$3\cdot\frac{2}{3}\int \frac{1\cdot\frac{1}{3}}{\sqrt{1-(\frac{x}{3})^2}}\,dx$

$u = \frac{1}{3}x$

$du = \frac{1}{3}dx$

$2\int \frac{1}{\sqrt{1-u^2}}\,du$

$2\sin^{-1}u + C$

$\boxed{2\sin^{-1}(\frac{1}{3}x)+C}$ ✓

20. $\int \tan^{-1} x\,dx$

$u = \tan^{-1}x$ $\quad v = x$

$du = \frac{1}{1+x^2}dx$ $\quad dv = dx$

$x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$u = 1+x^2$

$du = 2x\,dx$

$x\tan^{-1}x - \frac{1}{2}\int \frac{1}{u}\,du$

$\boxed{x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C}$ ✓

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21. $\int_1^{\boxed{2}} \frac{dx}{x-2}$

$\lim_{R\to 2^-} \int_1^R \frac{1}{x-2}\,dx$

$\lim_{R\to 2^-} \ln|x-2| \Big|_1^R$

$\lim_{R\to 2^-} \ln|R-2| - \ln|1-2|$

$\lim_{R\to 2^-} \ln|R-2| - \cancel{\ln(1)}$ ↙0

$\ln|1.999-2|$

$\ln(.001)$

$= -\infty$ ✓

$\boxed{\text{does not converge}}$

22. $\int_{-\infty}^{\infty} e^{2x}dx = \int_{-\infty}^{0} e^{2x} + \int_0^{\infty} e^{2x}$

$\lim_{R\to -\infty} \int_R^0 e^{2x}dx + \lim_{R\to\infty} \int_0^R e^{2x}dx$

$\lim_{R\to -\infty} \frac{1}{2}e^{2x}\Big|_R^0$ $\lim_{R\to\infty} \frac{1}{2}e^{2x}\Big|_0^R$

$\lim_{R\to -\infty} \frac{1}{2}e^0 - e^{2R}$ $\lim_{R\to\infty} \frac{1}{2}e^{2R} - \frac{1}{2}e^0$

$\frac{1}{2}(1) - e^{2(-\infty)}$ $\frac{1}{e}e^{\infty} - \frac{1}{2}$

$\frac{1}{2} - \cancel{\frac{1}{e^{200}}}$ ↙0

$\frac{1}{2}$ $\infty - \frac{1}{2} = \infty$

$\boxed{\text{does not converge}}$ ✓

23. $\int_{-\infty}^{-1} \frac{dx}{x^2}$

$\lim_{R\to -\infty} \int_R^{-1} x^{-2}dx$

$\lim_{R\to -\infty} \frac{x^{-1}}{-1}\Big|_R^1$

$\lim_{R\to -\infty} -\frac{1}{x}\Big|_R^1$

$\lim_{R\to -\infty} \frac{-1}{-1} - -\frac{1}{R}$

$\lim_{R\to -\infty} +1 + \cancel{\frac{1}{R}}$ ↙0 $= \boxed{+1}$ ✓

24. $\int_0^{\boxed{1}} \frac{dx}{1-x}$

$\lim_{R\to 1^-} -\int_1^R \frac{-1}{1-x}\,dx$ $u=1-x$, $du=-dx$

$\lim_{R\to 1^-} -\int_1^R \frac{1}{u}\,du$

$\lim_{R\to 1^-} -\ln|1-x|\Big|_0^R$

$\lim_{R\to 1^-} -\ln|1-R| - -\ln|1-0|$

$\lim_{R\to 1^-} -\ln|1-R|$

$-\ln|1-.9999|$

$-\ln(.00001)$ ✓

$-(-\infty) = \infty$

$\boxed{\text{does not converge}}$