Notes: Optimization — Day 6 Application of Derivatives

How To Solve

1. Assign symbols to all given quantities and quantities to be determined. When feasible, make a sketch.
2. Write a primary equation for the quantity that is to be maximized (or minimized).
3. Reduce the primary equation to one having a single variable. This may involve a secondary equation.
4. Take a derivative and set the derivative equal to zero. Solve for the variable.
5. Verify that what you have is a max. or min.

> What makes a problem an optimization problem? What are the steps to solve?

x & y are 2 numbers

Example 1: Find two positive numbers that satisfy the give requirements. The product is 240 and the sum is a minimum.

$\boxed{\text{verified minimum}}$

$P = 240$

$x \cdot y = 240$

$y = \frac{240}{x} = 240x^{-1}$

Primary

$S = x + y$

$S = x + 240x^{-1}$

$S' = 1 - 240x^{-2}$

$0 = 1 - \frac{240}{x^2}$

$\frac{240}{x^2} = \frac{1}{1}$

$x^2 = 240$ $\quad \boxed{x = \sqrt{240}}$

$-$ $\sqrt{240}$ (min) $+$

$S'(\sqrt{239})$ $\quad S'(\sqrt{241})$

$y' = s' = 1 - 240 \div x \wedge 2$

$y = \frac{240}{x} = \frac{240}{\sqrt{240}} \cdot \frac{\sqrt{240}}{\sqrt{240}}$ $\quad \boxed{y = \sqrt{240}}$

Example 2: Find the dimensions of a rectangle with perimeter 400 m whose area is as large as possible.

$P = 400$

$400 = 2l + 2w$

$2l = 400 - 2w$

$l = 200 - w$

Primary

$A = l \cdot w$

$A = (200 - w)w$

$A = 200w - w^2$

$A' = 200 - 2w$

$0 = 200 - 2w$

$2w = 200$

$\boxed{w = 100}$

verify

$+$ (max $w = 100$) $-$

$A'(99)$ $\quad A'(101)$

$l = 200 - w$

$l = 200 - 100$

$\boxed{l = 100}$

Example 3: Which points on the graph of $y = x^2 - 9$ are closest to the point (0,3)?

distance formula you need 2 points

$\boxed{1}$ $(0, 3)$

$\boxed{2}$ $(x, x^2 - 9)$

Primary

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{(x-0)^2 + (x^2 - 9 - 3)^2}$

$d = \sqrt{x^2 + (x^2 - 12)^2}$

$d = \sqrt{x^2 + x^4 - 24x^2 + 144}$

$d = (x^4 - 23x^2 + 144)^{1/2}$

$d' = \frac{1}{2}(x^4 - 23x^2 + 144)^{-1/2}(4x^3 - 46x)$

$d' = \frac{4x^3 - 46x}{2\sqrt{x^4 - 23x^2 + 144}}$

For optimization problems you only need to find where the derivative = 0. You do NOT need derivative = ud

$4x^3 - 46x = 0$

$2x(2x^2 - 23) = 0$

$2x = 0$ & $2x^2 - 23 = 0$

$x = 0$ & $x = \pm\sqrt{\frac{23}{2}} \approx \pm 3.4$

$\boxed{\text{verify}}$

$-$ $-\sqrt{\frac{23}{2}}$ (min) $+$ $\quad$ (max) 0 $\quad$ $-$ $\sqrt{\frac{23}{2}}$ (min) $+$

$d'(-4)$ $\quad d'(-1)$ $\quad d'(1)$ $\quad d'(4)$

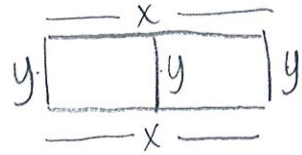
$x = \pm\sqrt{23/2}$

$\left(\pm\sqrt{\frac{23}{2}}, \frac{5}{2}\right)$

$y = x^2 - 9$

$y = \left(\pm\sqrt{\frac{23}{2}}\right)^2 - 9 = \frac{23}{2} - \frac{18}{2} = \frac{5}{2}$

Notes: Optimization



Day 6 Application of Derivatives

Example 4: A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

$A = 1,500,000$

$xy = 1,500,000$

$y = \frac{1,500,000}{x}$

$y = 1,500,000x^{-1}$

Cost = perimeter of rect.

$\text{Cost} = 2x + 3y$

$\text{Cost} = 2x + 3(1,500,000x^{-1})$

$\text{Cost} = 2x + 4,500,000x^{-1}$

$\text{Cost}' = 2 - 4,500,000x^{-2}$

$0 = 2 - \frac{4,500,000}{x^2}$

$\frac{4,500,000}{x^2} = \frac{2}{1}$

$2x^2 = 4,500,000$

$x^2 = 2,250,000$

$\boxed{x = 1500}$

1500, − min +

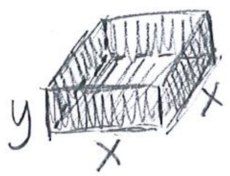
$C'(1499)$ $C'(1501)$

$y = \frac{1,500,000}{x}$

$y = \frac{1,500,000}{1500}$

$\boxed{y = 1000}$

Example 5: A Manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?



$SA = 108$

$x^2 + 4xy = 108$

A bottom + A(4 sides)

$\frac{4xy}{4x} = \frac{108 - x^2}{4x}$

Primary

$V = x^2y$

$V = x^2\left(\frac{108}{4x} - \frac{x^2}{4x}\right)$

$V = \frac{108x^2}{4x} - \frac{x^4}{4x}$

$V = 27x - \frac{1}{4}x^3$

$V' = 27 - \frac{3}{4}x^2$

$0 = 27 - \frac{3}{4}x^2$

$\frac{3}{4}x^2 = \frac{27}{1}$

$3x^2 = 108$

$x^2 = 36$

$\boxed{x = 6}$

$\boxed{\text{Verify}}$

+ 6 max −

$V'(5)$ $V'(7)$

$y = \frac{108}{4x} - \frac{x^2}{4x}$

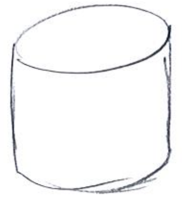
$y = \frac{108}{4(6)} - \frac{6}{4}$

$\boxed{y = 3}$

$d = l \cdot w \cdot h$

$d = 6 \cdot 6 \cdot 3$

Example 6: Find the minimum cost to construct a cylindrical container if the material for the top and bottom costs 4 cents per square inch and the material for the sides costs 3 cents per square inch. The container is to have a volume of 100 cubic inches.



$V = 100$

$\frac{\pi R^2 h}{\pi R^2} = \frac{100}{\pi R^2}$

$h = \frac{100}{\pi R^2}$

Primary

$\text{Cost} = .04\left(\begin{matrix}\text{Area}\\ \text{top \& bottom}\end{matrix}\right) + .03\left(\begin{matrix}\text{Area}\\ \text{side}\end{matrix}\right)$

$\text{Cost} = .04(2) \cdot \pi R^2 + .03(h \cdot 2\pi R)$

$\text{Cost} = .08\pi R^2 + .06\pi Rh$

$\text{Cost} = .08\pi R^2 + .06\cancel{\pi} R\left(\frac{100}{\cancel{\pi} R^2}\right)$

$\text{Cost} = .08\pi R^2 + 6R^{-1}$

$\text{Cost}' = .16\pi R - 6R^{-2}$

$0 = .16\pi R - \frac{6}{R^2}$

$\frac{6}{R^2} = \frac{.16\pi R}{1}$

$6 = .16\pi R^3$

$R^3 = \frac{6}{.16\pi}$

$R = \sqrt[3]{\frac{6}{.16\pi}}$

$R = 2.28539$

$\boxed{\text{verify}}$

− 2.285 Min +

$C'(2)$ $C'(3)$

$\text{Cost} = .08\pi R^2 + \frac{6}{R}$

$\boxed{\text{Cost}(2.28539) = \$3.94}$