Notes: Derivatives Of Inverse Functions

Remember:

How do you find the inverse of a function?

Easy as 1→ 2→ 3

1. Rewrite $f(x)$ as $y=$
2. Switch x & y
3. Solve for y

$\boxed{f(x)=e^x}$

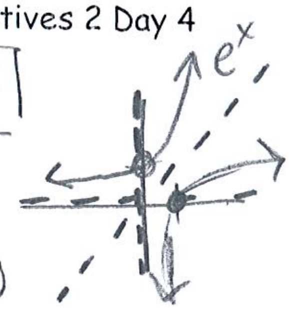
$y=e^x$

$x=e^y$

$\ln x=\ln e^y$

$y=\ln x$

$\boxed{f^{-1}(x)=\ln x}$



Example 1: Find the inverse of $\boxed{f(x)=\frac{3x}{x+4}}$

$y=\frac{3x}{x+4}$

$\frac{x}{1}=\frac{3y}{y+4}$

$x(y+4)=1(3y)$

$xy+4x=3y$

$xy-3y=-4x$

$y(x-3)=-4x$

$y=\frac{-4x}{x-3}$

$f^{-1}(x)=\boxed{\frac{-4x}{x-3}}$

Thm.

Derivative of Inverse Functions:

Let $f(x)$ be some function

Let $f^{-1}(x)$ be the inverse

Then $\boxed{\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{f'[f^{-1}(x)]}}$

D
Derivative
Of
Inverse

Example 2: Find the Derivative of the Inverse given $f(x)=x^2+4$ $d[x\geq 0]$

A. Find the inverse and then take derivative

$f(x)=x^2+4$

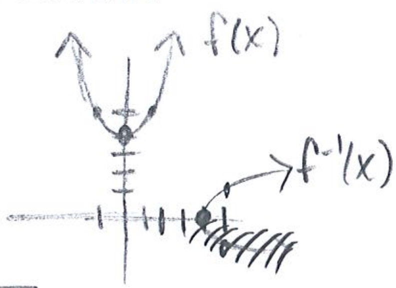
$y=x^2+4$

$x=y^2+4$

$\sqrt{y^2}=\sqrt{x-4}$

$y=\pm\sqrt{x-4}$

$\boxed{f^{-1}(x)=\sqrt{x-4}}$



$\frac{d}{dx}\left[f^{-1}(x)=(x-4)^{1/2}\right]$

$\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{2}(x-4)^{-1/2}(1)$

$\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{2\sqrt{x-4}}$

B. Use the Thm. to find the derivative of the inverse.

Know: $\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{f'[f^{-1}(x)]}$

$\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{f'[\sqrt{x-4}]}$

$f(x)=x^2+4$

$f'(x)=2x$

$f'(\sqrt{x-4})=2\sqrt{x-4}$

$\therefore \frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{2\sqrt{x-4}}$

Example 3: Calculate the derivative of the inverse **without** finding the inverse function.

Find $\frac{d}{dx}\left[f^{-1}(x)\right]_{x=1}$ if $f(x)=x+e^x$

Know: $\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{f'\left[f^{-1}(x)\right]}$

$f^{-1}(1)=\underline{0}$

$f(0)=1$

$f(x)=x+e^x$

$1=x+e^x$ let $x=0$ $1=0+e^0$

$1=1$

$f(x)=x+e^x$

$f'(x)=1+e^x$

$f'(0)=1+e^0=2$

$=\frac{1}{f'\left[f^{-1}(1)\right]}$

$=\frac{1}{f'[0]}$

$=\boxed{\frac{1}{2}}$

Example 4:

A. Find $(f^{-1})'(3)$ if $f(x)=2x^3-3x+3$

B. Find $(f^{-1})'(2)$ if $f(x)=2x^3-3x+3$

Know: $\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{f'\left[f^{-1}(x)\right]}$

$f^{-1}(3)=\underline{0}$ & $f^{-1}(2)=\underline{1}$

$f(0)=3$ $\quad f(1)=2$

let $x=0$ $2(0)^3-3(0)+3$

$y=3$

let $x=1$ $2(1)^3\cancel{-3(1)+3}$

$y=2$

$f(x)=2x^3-3x+3 \rightarrow f'(0)=-3$

$f'(x)=6x^2-3 \longrightarrow f'(1)=3$

$=\frac{1}{f'\left[f^{-1}(3)\right]}$

$=\frac{1}{f'[0]}$

$=\boxed{\frac{1}{-3}}$

$=\frac{1}{f'\left[f^{-1}(2)\right]}$

$=\frac{1}{f'[1]}$

$=\boxed{\frac{1}{3}}$

Example 5: Find $\frac{d}{dx}\left[f^{-1}(x)\right]_{x=-1}$ if $f(x)=x^3-9$

Know: $\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{f'\left[f^{-1}(x)\right]}$

$f^{-1}(-1)=\underline{2}$

let $x=2$

$y=2^3-9$

$y=8-9=-1$

$f(2)=-1$

$f(x)=x^3-9$

$f'(x)=3x^2$

$f'(2)=3(2)^2=12$

$=\frac{1}{f'\left[f^{-1}(-1)\right]}$

$=\frac{1}{f'[2]}$

$=\boxed{\frac{1}{12}}$

$\frac{d}{dx}[sin^{-1}(AT)] = \frac{1}{\sqrt{1-(AT)^2}} \cdot \frac{d}{dx}[AT]$

$\frac{d}{dx}[cos^{-1}(AT)] = \frac{-1}{\sqrt{1-(AT)^2}} \cdot \frac{d}{dx}[AT]$

$\frac{d}{dx}[tan^{-1}(AT)] = \frac{1}{(AT)^2+1} \cdot \frac{d}{dx}[AT]$

$\frac{d}{dx}[cot^{-1}(AT)] = \frac{-1}{(AT)^2+1} \cdot \frac{d}{dx}[AT]$

$\frac{d}{dx}[sec^{-1}(AT)] = \frac{1}{|AT|\sqrt{(AT)^2-1}} \cdot \frac{d}{dx}[AT]$

$\frac{d}{dx}[csc^{-1}(AT)] = \frac{-1}{|AT|\sqrt{(AT)^2-1}} \cdot \frac{d}{dx}[AT]$

D $\frac{d}{dx}[sin^{-1}(AT)] =$	D $\frac{d}{dx}[cos^{-1}(AT)] =$	D $\frac{d}{dx}[tan^{-1}(AT)] =$
D $\frac{d}{dx}[cot^{-1}(AT)] =$	D $\frac{d}{dx}[sec^{-1}(AT)] =$	D $\frac{d}{dx}[csc^{-1}(AT)] =$

Example 6: Calculate $\frac{d}{dx}[tan^{-1}(3x+1)]$

Know: $\frac{d}{dx}[tan^{-1}(AT)] = \frac{1}{(AT)^2+1} \cdot \frac{d}{dx}[AT]$

$= \frac{1}{(3x+1)^2+1} \cdot \frac{d}{dx}[3x+1]$

FRQ ✓ $= \frac{3}{9x^2+6x+1+1}$

$= \frac{3}{9x^2+6x+2}$

Example 7: Calculate $\frac{d}{dx}[csc^{-1}(e^x+1)|_{x=0}]$

Know: $\frac{d}{dx}[csc^{-1}(AT)] = \frac{-1}{|AT|\sqrt{(AT)^2-1}} \cdot AT'$

$= \frac{-1}{|e^x+1|\sqrt{(e^x+1)^2-1}} \cdot \frac{d}{dx}[e^x+1]$

$= \frac{-e^x}{|e^x+1|\sqrt{(e^x+1)^2-1}} = \frac{-e^0}{|e^0+1|\sqrt{(e^0+1)^2-1}}$

$= \frac{-1}{2\sqrt{3}}$

Example 8: $\frac{d}{dx}[sin^{-1}(12x)]$

$\frac{1}{\sqrt{1-(AT)^2}} \cdot \frac{d}{dx}[AT] = \frac{d}{dx}[sin^{-1}(AT)]$

$\frac{1}{\sqrt{1-(12x)^2}} \cdot \frac{d}{dx}[12x]$

$\frac{12}{\sqrt{1-144x^2}}$

Example 9: $f(x) = \arcsin(2x)$ Find $f'(2)$

Example 10:

$f(x) = \left(\cos^{-1}(x^2)\right)^3$ Find $f'(x)$

Example 11:

$f(x) = 2x\cos^{-1}\left(5x^2\right)$ Find $f'(x)$

Example 12:

$f(x) = \sqrt{1-x^2}\arcsin x$ Find $f'(x)$

Example 13: $\frac{d}{dx}\left[tan\left(\frac{1+x}{1-x}\right)\right]$