BC Calculus

Models For Population Growth

Name________________________ Pd.___

Additional Differential Equations Day 4

1. The population of the world was about 5.3 billion in 1990. Birth rates in the 1900s ranged from 35 to 40 million per year and death rates ranged from 15 to 20 million per year. Let's assume that the carrying capacity for the world population is 100 billion.

A.) Write the logistic differential equation for these data. (Because the initial population is small compared to the carrying capacity, you can take k to be an estimate of the initial relative growth rate.)

$$k \approx \frac{1}{P}\frac{dP}{dt} \approx \frac{1}{5.3}(.02) \approx \frac{1}{265}$$

B.) Use the logistic model to estimate the world population in the year 2000 and compare with the actual population of 6.1 billion.

Year 2000 = 10 years since 1990 $\therefore t = 10$

$$C = \frac{5.3}{5.3-100} = \frac{-53}{947} \qquad P(t) = \frac{100}{1-\frac{e^{-1/265\,t}}{-53/947}} = \frac{100}{\frac{1+947e^{-\frac{1}{265}t}}{53}} = \frac{5300}{53+947e^{-1/265\,t}}$$

$$P(10) = \frac{5300}{53+947e^{-1/265(10)}} = 5.493 \text{ billion}$$

C.) Use the logistic model to predict the world population in the years 2100 and 2500.

$t = 110 \qquad t = 510$

$P(110) = 7.814$ billion

$P(510) = 27.718$ billion

D.) What are your predictions if the carrying capacity is 50 billion? $A = 50$

$$C = \frac{5.3}{5.3-50} = \frac{-53}{447} \qquad P(t) = \frac{50}{1-\frac{e^{-1/265\,t}}{-53/447}} = \frac{50}{\frac{1+447e^{-1/265\,t}}{53}} = \frac{2650}{53+447e^{-1/265\,t}}$$

$P(110) = 7.612$ Billion

$P(510) = 22.413$ Billion

2. Suppose that an initial population of 10,000 bacteria grows exponentially at a rate of 2% per hour and that $y = y(t)$ is the number of bacteria present t hours later.

A.) Find a differential equation whose solution is y(t).

$$\frac{dy}{dt} = ky \qquad \frac{dy}{dt} = .02y$$

B.) Find a formula for y(t), using the initial condition.

$$y(t) = 10000e^{.02t}$$

C.) How long does it take for the initial population of bacteria to double?

$$20000 = 10000e^{.02t}$$

$$2 = e^{.02t}$$

$$.02t = \ln(2)$$

$$t = \frac{\ln(2)}{.02}$$

$t = 34.657$ hours

2. Suppose that 100 fruit flies are placed in a breeding container that can support at most 10,000 flies. Assuming that the population grows exponentially at a rate of 2% per day, how long will it take for the container to reach capacity?

$$y = 100e^{.02t}$$

$$10000 = 100e^{.02t}$$

$$100 = e^{.02t}$$

$$.02t = \ln 100$$

$$t = \frac{\ln 100}{.02} = 230.258 \text{ days}$$

3. A bacteria population grows at a rate proportional to its size. The initial count was 400 after and after 4 hours, there were 25,000. In how many *minutes* does the population double?

$$y = P_0 e^{kt}$$

$$y = 400e^{kt}$$

$$25000 = 400e^{k(4)}$$

$$\frac{25000}{400} = e^{4k}$$

$$k = \frac{\ln\left(\frac{250}{4}\right)}{4}$$

$$K = 1.034$$

$$y = 400e^{1.034t}$$

$$800 = 400e^{1.034t}$$

$$2 = e^{1.034t}$$

$$t = \frac{\ln 2}{1.034}$$

$$t = .67 \text{ hours}$$

$$\boxed{t = 40.22 \text{ min}}$$

4. Forest Service ecologists have determined that a national forest can sustain a maximum population of 900 deer. When the forest is first designated as a wildlife preserve there are 100 deer. Two years later, there are 140 deer.

A.) Find $\lim_{t\to\infty} P(t)$. $= 900$

B.) Find the differential equation that models the deer population.

$$\frac{dy}{dt} = Ky\left(1 - \frac{y}{900}\right)$$

$$\frac{dy}{dt} = \frac{1}{5}y\left(1 - \frac{y}{900}\right)$$

$$K = \frac{1}{Y_0} \cdot \frac{dy}{dt} = \frac{1}{100}\left(\frac{140-100}{2-0}\right)$$

$$= \frac{1}{100}\left(\frac{40}{2}\right) = \frac{1}{5}$$

C.) Find the equation P(t) that models the deer population.

$$P(t) = \frac{A}{1 - \frac{e^{-Kt}}{C}} = \frac{900}{1 - \frac{e^{-1/5t}}{C}}$$

$$C = \frac{100}{100 - 900} = -\frac{1}{8}$$

$$= \frac{900}{1 - \frac{e^{-1/5t}}{-1/8}} = \frac{900}{1 + 8e^{-1/5t}}$$