AP Calculus-AB

Notes: Formal Definition of Derivative

Terminology and Notation:

The *derivative* of f at $x=a$ is the instantaneous rate of change of f at $x=a$. The following notations are used for the derivative of f at $x=a$:

$f'(a)$ $\qquad$ $\left.\dfrac{df}{dx}\right|_{x=a}$

Derivative: $f'(x)$	Derivative at a point: $f'(a)$ where a is some number	
$\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$	$\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$	$\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

$f'(x) \rightarrow$ slope at all points

$f'(\text{some } \#) \rightarrow$ slope at one pt.

Example(s) 1:

A.) Find $\dfrac{dy}{dx}$ for $f(x)=x^2+1$

same as find $f'(x)$

$f(x)=x^2+1$

$f(x+h)=(x+h)^2+1$

$=1(x)^2(h)^0+2(x)^1(h)^1+1(x)^0(h)^2+1$

$=x^2+2xh+h^2+1$

$1 \quad (x+h)^0$

$1 \quad 1 \quad (x+h)^1$

$1 \quad 2 \quad 1 \quad (x+h)^2$

$1 \quad 3 \quad 3 \quad 1 \quad (x+h)^3$

$f'(x)=\lim\limits_{h\to 0}\dfrac{\boxed{f(x+h)}-\boxed{f(x)}}{h}$

$f'(x)=\lim\limits_{h\to 0}\dfrac{x^2+2xh+h^2+1-(x^2+1)}{h}$

$f'(x)=\lim\limits_{h\to 0}\dfrac{\cancel{x^2}+2xh+h^2\cancel{+1}\cancel{-x^2}\cancel{-1}}{h}$

$f'(x)=\lim\limits_{h\to 0}\dfrac{\cancel{h}(2x+h)}{\cancel{h}}$

$f'(x)=2x+0$

$f'(x)=2x$

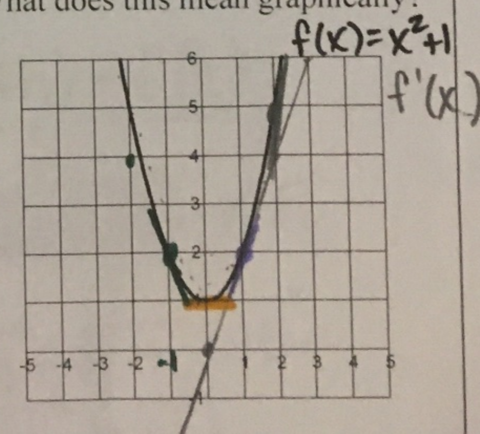
B.) Find $\left.\dfrac{dy}{dx}\right|_{x=-1}$ $\quad f'(-1)=2(-1)=-2$

C.) Find $\left.\dfrac{dy}{dx}\right|_{x=0}$ $\quad f'(0)=2(0)=0$

D.) Find $\left.\dfrac{dy}{dx}\right|_{x=1}$ $\quad f'(1)=2(1)=2$

E.) Find $\left.\dfrac{dy}{dx}\right|_{x=2}$ $\quad f'(2)=2(2)=4$

What does this mean graphically?



Example 2:

Find $f'(x)$ for $f(x)=\sqrt{x-2}$

$f(x)=\sqrt{x-2}$

$f(x+h)=\sqrt{x+h-2}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{(\sqrt{x+h-2}-\sqrt{x-2})(\sqrt{x+h-2}+\sqrt{x-2})}{(h)(\sqrt{x+h-2}+\sqrt{x-2})}$$

$$f'(x)=\lim_{h\to 0}\frac{\cancel{x}+h-\cancel{2}-\cancel{(x-2)}}{h(\sqrt{x+h-2}+\sqrt{x-2})}\quad -\cancel{x}+\cancel{2}$$

$$f'(x)=\lim_{h\to 0}\frac{\cancel{h}\cdot 1}{\cancel{h}(\sqrt{x+h-2}+\sqrt{x-2})}$$

$$f'(x)=\frac{1}{\sqrt{x-2}+\sqrt{x-2}}=\frac{1}{2\sqrt{x-2}}$$

Example 3:

Remember: $f'(x)$ is the same as instant rate of change.

Find $f'(x)$ for $f(x)=x^3-2x$

$f(x)=x^3-2x$

$f(x+h)=(x+h)^3-2(x+h)$

$=1\cdot x^3\cdot h^0+3\cdot x^2\cdot h^1+3x^1h^2+1\cdot x^0\cdot h^3-2x-2h$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{2x}-2h-\cancel{(x^3-2x)}}{h}\quad -\cancel{x^3}+\cancel{2x}$$

$$f'(x)=\lim_{h\to 0}\frac{\cancel{h}(3x^2+3xh+h^2-2)}{\cancel{h}}$$

$$f'(x)=3x^2+3x(0)+(0)^2-2$$

$1\ (x+h)^0$

$1\ 1\ (x+h)^1$

$1\ 2\ 1\ (x+h)^2$

$1\ 3\ 3\ 1\ (x+h)^3$

$$f'(x)=3x^2-2$$

Example 4:

Find the instant rate of change of the function at $x=2$

$f(t)=6+\sqrt{t}$

$f(t)=6+\sqrt{t}$

$f(t+h)=6+\sqrt{t+h}$

$$f'(t)=\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

$$f'(t)=\lim_{h\to 0}\frac{\cancel{6}+\sqrt{t+h}-\cancel{(6+\sqrt{t})}}{h}\quad -\cancel{6}-\sqrt{t}$$

$$f'(t)=\lim_{h\to 0}\frac{(\sqrt{t+h}-\sqrt{t})(\sqrt{t+h}+\sqrt{t})}{h(\sqrt{t+h}+\sqrt{t})}$$

$$f'(t)=\lim_{h\to 0}\frac{(\cancel{t}+\cancel{h}-\cancel{t})1}{\cancel{h}(\sqrt{t+h}+\sqrt{t})}\qquad f'(t)=\frac{1}{2\sqrt{t}}$$

Tangent Lines: You will always need two things for a tangent line:

1. Point (x, y) OR $f(x) = y$
2. Slope m OR $f'(x)$

There are three forms of a line:

1. ~~Slope/Intercept~~ (y) $y = mx + b$
2. ~~Point/Slope~~ $y - y_1 = m(x - x_1)$
3. General $ax + by = c$
   - ax term is positive
   - equation has no fractions

Example 5:

Find the equation for the tangent line to $g(x) = \frac{8}{x}$ at $x = 4$.

Tangent Line

$\boxed{1}$ Point: $g(4) = 2$

$\boxed{2}$ Slope: $g'(4) = \frac{-1}{2}$

$g(x) = \frac{8}{x}$

$g(4) = \frac{8}{4} = 2$

$g(4+h) = \frac{8}{4+h}$

$y - 2 = -\frac{1}{2}(x - 4)$

$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

$g'(4) = \lim_{h \to 0} \frac{g(4+h) - g(4)}{h}$

$g'(4) = \lim_{h \to 0} \frac{\frac{8}{4+h} - \frac{2}{1}\frac{(4+h)}{(4+h)}}{h}$

$g'(4) = \lim_{h \to 0} \frac{\frac{\cancel{8} - \cancel{8} - 2h}{4+h}}{h}$

$g'(4) = \lim_{h \to 0} \frac{\frac{-2\cancel{h}}{4+h} \cdot \frac{1}{\cancel{h}}}{\frac{\cancel{h}}{1} \cdot \frac{1}{\cancel{h}}}$ $g'(4) = \frac{-2}{4+0} = -\frac{1}{2}$

Example 6:

Find the equation for the tangent line to $g(x) = 2x - 3$ at $x = 2$.

Tangent Line

$\boxed{1}$ Point: $g(2) = 1$

$\boxed{2}$ Slope: $g'(2) = 2$

$g(x) = 2x - 3$

$g(2) = 2(2) - 3 = 1$

$g(2+h) = 2(2+h) - 3$

$= 4 + 2h - 3$

$2h + 1$

$y - 1 = 2(x - 2)$

$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

$g'(2) = \lim_{h \to 0} \frac{g(2+h) - g(2)}{h}$

$g'(2) = \lim_{h \to 0} \frac{2h \cancel{+1} \cancel{-1}}{h}$

$g'(2) = \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}} = \lim_{h \to 0} 2$

$g'(2) = 2$

Memorize each of these!!!

Derivative: $f'(x)$	Derivative at a point: $f'(a)$ where a is some number	
$\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$	$\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}$	$\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}$

Example(s) 7:

What is $f(x)$ in each of the following? What is each question asking you to answer?

$\lim\limits_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = f'(x)$ $f(x)=\sqrt{x}$	$\lim\limits_{h\to 0}\frac{\sqrt{\cancel{4+h}}-\sqrt{4}}{h}$ or $\lim\limits_{h\to 0}\frac{\sqrt{4+h}-2}{h}$ ✓ $= f'(4)$ $f(x)=\sqrt{x}$	$\lim\limits_{x\to 4}\frac{\sqrt{x}-\sqrt{4}}{x-4}$ or $\lim\limits_{x\to 4}\frac{\sqrt{x}-2}{x-4} = f'(4)$ $f(x)=\sqrt{x}$
$\lim\limits_{h\to 0}\frac{(x+h)^5-x^5}{h} = f'(x)$ $f(x)=x^5$	$\lim\limits_{h\to 0}\frac{(\cancel{2+h})^5-32}{h} = f'(2)$ $f(x)=x^5$	$\lim\limits_{x\to 2}\frac{x^5-32}{x-2} = f'(2)$ $f(x)=x^5$
$\lim\limits_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = f'(x)$ $f(x)=\cos x$	$\lim\limits_{h\to 0}\frac{\cos(\cancel{\pi+h})-(-1)}{h} = f'(\pi)$ $f(x)=\cos x$	$\lim\limits_{x\to \pi}\frac{\cos(x)-(-1)}{x-\pi} = f'(\pi)$ $f(x)=\cos x$
$\lim\limits_{x\to -1}\frac{(x^2+2)-(3)}{x+1} = f'(-1)$ $f(x)=x^2+2$	$\lim\limits_{h\to 0}\frac{(x+h)^2+2-(x^2+2)}{h} = f'(x)$ $f(x)=x^2+2$	$\lim\limits_{h\to 0}\frac{((\cancel{-1+h})^2+2)-(3)}{h} = f'(-1)$ $f(x)=x^2+2$
$\lim\limits_{h\to 0}\frac{\frac{1}{2+h}-\frac{1}{2}}{h} = f'(2)$ $f(x)=\frac{1}{x}$	$\lim\limits_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2} = f'(2)$ $f(x)=\frac{1}{x}$	$\lim\limits_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = f'(x)$ $f(x)=\frac{1}{x}$
$\lim\limits_{h\to 0}\frac{\left[(x+h)^3-2\right]-(x^3-2)}{h} = f'(x)$ $f(x)=x^3-2$	$\lim\limits_{x\to \frac{\pi}{4}}\frac{\tan x-1}{x-\frac{\pi}{4}} = f'(\frac{\pi}{4})$ $f(x)=\tan x$	$\lim\limits_{h\to 0}\frac{\ln(\cancel{5+h})-\ln 5}{h} = f'(5)$ $f(x)=\ln x$
$\lim\limits_{x\to \frac{\pi}{2}}\frac{\sec x-1}{x-\frac{\pi}{2}} = f'(\frac{\pi}{2})$ $f(x)=\sec x$	$\lim\limits_{h\to 0}\frac{\cancel{e^{2+h}}-e^2}{h} = f'(2)$ $f(x)=e^x$	$\lim\limits_{h\to 0}\frac{\left[\lvert x+h\rvert-1\right]-(\lvert x\rvert-1)}{h} = f'(x)$ $f(x)=\lvert x\rvert-1$