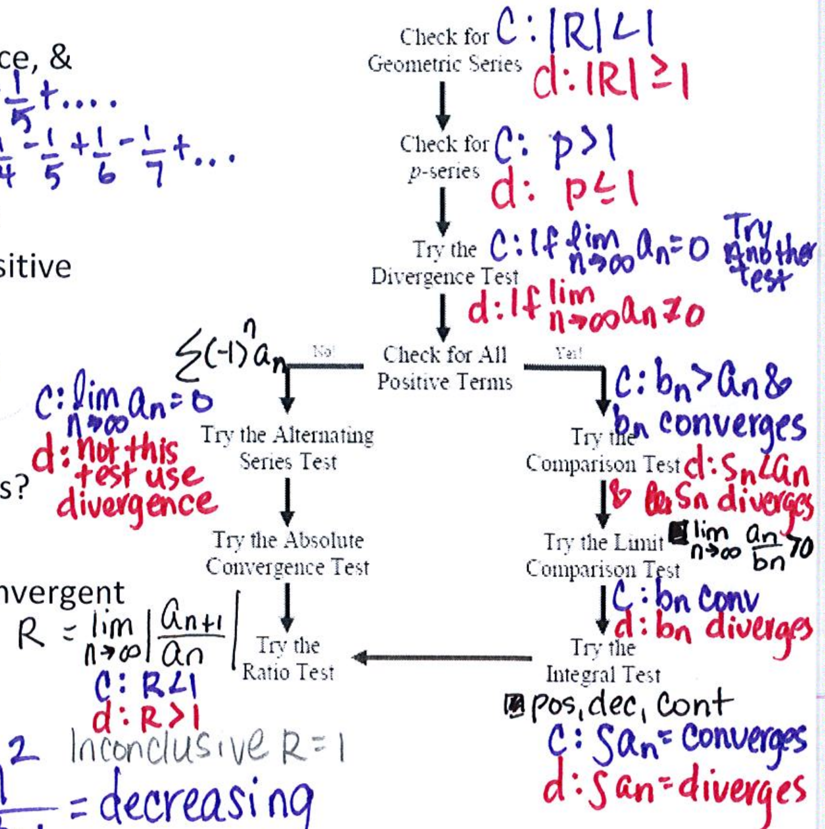
Notes: Alternating Series, Absolute Convergence, & Conditional Convergence $\sum \frac{1}{n} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots$

Infinite Series Day 6 $\sum(-1)^n\left(\frac{1}{n}\right) = -\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\dots$



Alternating Series Test: The alternating series $\sum(-1)^n a_n$, where a_n is a sequence with all positive terms,

Converges: If a_n is decreasing and $\lim_{n\to\infty} a_n = 0$

What does the Alternating Series test not tell us?

diverges!

Determine whether the following series are convergent or divergent. Justify your answer.

Example One:

$\sum_{n=3}^{\infty} \frac{(-1)^{n+1}n^2}{n^3+1}$ Alternating

$a_n = \frac{n^2}{n^3+1}$ = decreasing

$\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$

$\therefore \sum_{n=3}^{\infty} \frac{(-1)^{n+1}n^2}{n^3+1}$ converges by alternating.

Example Two:

$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{2n^2}{n^3+4}$ alternating

$\lim_{n\to\infty} \frac{2n^2}{n^3+4} = 0$

$\therefore \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2n^2}{n^3+4}$ converges by alternating

Example Three:

$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n\ln(n)}$ alternating

$\lim_{n\to\infty} \frac{1}{n\cdot\ln(n)} = 0$ $\therefore \sum_{n=0}^{\infty} \frac{(-1)^n}{n\cdot\ln(n)}$ converges by alternating

Review of Factorals

$$\frac{5!}{7!} = \frac{\cancel{5!}}{7 \cdot 6 \cdot \cancel{5!}} = \frac{1}{42}$$

$$\frac{n!}{(n+2)!} = \frac{\cancel{n!}}{(n+2)(n+1)\cancel{n!}} = \frac{1}{(n+2)(n+1)}$$

Ratio Test: Let $\{a_n\}$ be a sequence and assume that the following limit exists:

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Converges: If $r < 1$ then $\sum a_n$ converges ABSOLUTELY

Diverges: If $r > 1$ then $\sum a_n$ diverges

Inconclusive: If $r = 1$

Determine whether the following series are convergent or divergent. Justify your answer.

Example Four:

$\sum_{n=1}^{\infty} \frac{1}{n!}$

$$R = \lim_{n\to\infty} \frac{\overbrace{\frac{1}{(n+1)!}}^{a_{n+1}}}{\underbrace{\frac{1}{n!}}_{a_n}} = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \lim_{n\to\infty} \frac{n!}{(n+1)!}$$

$$R = \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n!}$ converges ABSOLUTELY by Ratio

$$2^A \cdot 2^B = 2^{A+B}$$

Example Five:

$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$$R = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{\cancel{2^n}(n+1)^2}{\cancel{2^n}\cdot 2^1 \cdot n^2}$$

$$R = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \frac{1}{2} < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges ABSOLUTELY by Ratio

Example Six:

$$\sum_{n=0}^{\infty}\left|(-1)^n \frac{n!}{100^n}\right| \quad R = \lim_{n\to\infty} \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} = \lim_{n\to\infty} \frac{\cancel{100^n}(n+1)\cancel{n!}}{\cancel{100^n}\cdot 100^1 \cdot \cancel{n!}}$$

$$R = \lim_{n\to\infty} \frac{n+1}{100} = \infty > 1$$

$\therefore \sum_{n=0}^{\infty} \frac{(-1)^n n!}{100^n}$ diverges by Ratio

Example Seven:

$$\sum_{n=0}^{\infty}\left|\frac{n^3}{(-3)^n}\right| \quad R = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{\cancel{3^n}(n+1)^3}{\cancel{3^n}\cdot 3(n^3)} = \frac{1}{3} < 1$$

$\therefore \sum_{n=0}^{\infty} \frac{n^3}{(-3)^n}$ converges by Ratio

Example Eight:

$$\sum_{n=1}^{\infty}\left|(-1)^{n+3} \frac{\sqrt{n}}{3n-2}\right| \quad R = \lim_{n\to\infty} \frac{\sqrt{n+1}}{3(n+1)-2} \cdot \frac{3n-2}{\sqrt{n}}$$

$$R = \lim_{n\to\infty} \frac{(3n-2)\sqrt{n+1}}{(3n+1)\sqrt{n}} = 1$$ Ratio does not work!

Alternating & $\lim_{n\to\infty} \frac{\sqrt{n}}{3n-2} = 0$

$\therefore \sum_{n=1}^{\infty} (-1)^{n+3} \frac{\sqrt{n}}{3n-2}$ converges by alternating

$$R = \lim_{n\to\infty} \frac{4^{2(n+1)+1}}{(n+1+1)10^{n+1}} \cdot \frac{(n+1)10^n}{4^{2n+1}}$$

Example Nine:

$$\sum_{n=0}^{\infty} \frac{4^{2n+1}}{(n+1)10^n}$$

$$R = \lim_{n\to\infty} \frac{4^{2n+2+1}}{(n+2)10^{n+1}} \cdot \frac{(n+1)10^n}{4^{2n+1}}$$

$$R = \lim_{n\to\infty} \frac{\cancel{4^{2n}} \cdot 4^{\cancel{3}\,2} \cdot \cancel{10^n}(n+1)}{\cancel{4^{2n}} \cdot \cancel{4^1} \cdot \cancel{10^n} \cdot 10^1(n+2)} = \lim_{n\to\infty} \frac{16(n+1)}{10(n+2)} = \frac{8}{5} > 1$$

$\therefore \sum_{n=0}^{\infty} \frac{4^{2n+1}}{(n+1)10^n}$ diverges by Ratio test.

Root Test: Let $\{a_n\}$ be a sequence and assume that the following limit exists:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

Converges: If $L < 1$ then $\sum a_n$ converges ABSOLUTELY

Diverges: If $L > 1$ then $\sum a_n$ diverges

Inconclusive: If $L = 1$

Example Ten:

$$\sum_{n=0}^{\infty} \left(\frac{n}{2n+3}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+3}\right)^n}$$

$$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} < 1$$

Converges absolutely by Root test