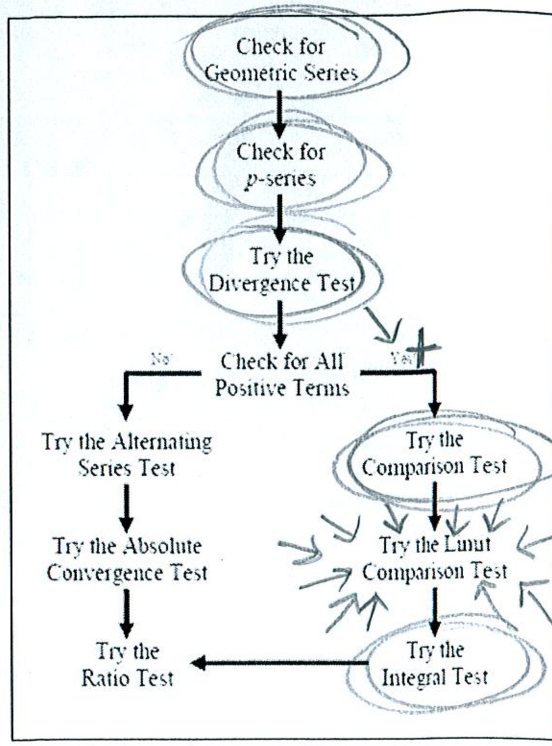
Day 5

Notes: Convergence of Series with Positive Terms (Continued)



Limit Comparison Test: Let a_n and b_n be positive sequences. Assume that the following limit exists:

$L = \lim\limits_{n\to\infty} \frac{a_n}{b_n}$ ← We pick b_n by comparing the degree in top & bottom.

If $L > 0$ *and* If $\sum_{n=1}^{\infty} b_n$ converges
Then $\sum_{n=1}^{\infty} a_n$ converges
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When is the limit comparison test inconclusive?
If the limit comparison test is inconclusive what does that mean you have to do to solve the problem?

When the $\lim\limits_{n\to\infty} \frac{a_n}{b_n} \le 0$, try a different test.

Example One: Show that $\sum_{n=1}^{\infty} \frac{n^2}{n^4 - n}$ converges.

$b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$ $\quad \lim\limits_{n\to\infty} \frac{\frac{n^2}{n^4-n} \cdot \frac{n^2}{1}}{\frac{1}{n^2} \cdot \frac{n^2}{1}} = \lim\limits_{n\to\infty} \frac{n^4}{n^4-n} = 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series w/ $p=2$ which is >1.

$\sum_{n=1}^{\infty} \frac{n^2}{n^4-n}$ converges by the limit comparison test b.c. $\lim\limits_{n\to\infty} \frac{\frac{n^2}{n^4-n}}{\frac{1}{n^2}} = 1 > 0$ And $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges.

Example Two: Determine whether $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2-4}}$ converges.

$b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n}$ $\quad \lim\limits_{n\to\infty} \frac{\frac{1}{\sqrt{n^2-4}} \cdot \frac{\sqrt{n^2}}{1}}{\frac{1}{\sqrt{n^2}} \cdot \frac{\sqrt{n^2}}{1}} = \lim\limits_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{n^2-4}} = 1$

$\sum_{n=3}^{\infty} \frac{1}{n}$ diverges by p-series $p=1$ which is ≤ 1.

$\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2-4}}$ diverges by the limit comparison test b.c. $\lim\limits_{n\to\infty} \frac{\frac{1}{\sqrt{n^2-4}}}{\frac{1}{\sqrt{n^2}}} = 1 > 0$ And $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges.

Example Three: Determine whether the following series are convergent or divergent. Justify your answer.

A. $\sum_{n=3}^{\infty} \frac{1}{4^n-3}$

$b_n = \frac{1}{4^n}$

$$\lim_{n\to\infty} \frac{\frac{1}{4^n-3}\cdot\frac{4^n}{1}}{\frac{1}{4^n}\cdot\frac{4^n}{1}} = \lim_{n\to\infty} \frac{\frac{d}{dn}[4^n]}{\frac{d}{dn}[4^n-3]} = \lim_{n\to\infty} \frac{\ln(4)\cdot 4^n}{\ln(4)\cdot 4^n} = \lim_{n\to\infty} 1 = 1$$

$\sum_{n=3}^{\infty} \frac{1}{4^n} = \sum_{n=3}^{\infty} \left(\frac{1}{4}\right)^n$ converges by geometric $|R| = \frac{1}{4} < 1$

$\sum_{n=3}^{\infty} \frac{1}{4^n-3}$ converges by the limit comparison test b.c. $\lim_{n\to\infty} \frac{\frac{1}{4^n-3}}{\frac{1}{4^n}} = 1 > 0$ And $\sum_{n=3}^{\infty} \frac{1}{4^n}$ converges.

B. $\sum_{n=1}^{\infty} \frac{3n^2+3}{n^3+n+1}$

$b_n = \frac{n^2}{n^3} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{3n^2+3}{n^3+n+1}\cdot\frac{n}{1}}{\frac{1}{n}\cdot\frac{n}{1}} = \lim_{n\to\infty} \frac{3n^3+3n}{n^3+n+1} = 3$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series b.c. $p = 1$ which is ≤ 1.

$\sum_{n=1}^{\infty} \frac{3n^2+3}{n^3+n+1}$ diverges by the limit comparison test b.c. $\lim_{n\to\infty} \frac{\frac{3n^2+3}{n^3+n+1}}{\frac{1}{n}} = 3 > 0$ And $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

C. ~~B.~~ $\sum_{n=1}^{\infty} \frac{\sqrt{n^3-12n-6}}{2n^2-4n-1}$

$\boxed{\text{Reminder } \sqrt[n]{x^m} = x^{\frac{m}{n}}}$

$(\sqrt{2})(\sqrt{3}) = \sqrt{6}$

$b_n = \frac{n^{3/2}}{n^2} = \frac{1}{n^{1/2}} = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^3-12n-6}}{2n^2-4n-1}\cdot\frac{\sqrt{n}}{1}}{\frac{1}{\sqrt{n}}\cdot\frac{\sqrt{n}}{1}} = \lim_{n\to\infty} \frac{\sqrt{n^4-12n^2-6n}}{2n^2-4n-1} = \frac{1}{2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by p-series b.c. $p = \frac{1}{2}$ which is ≤ 1.

$\sum_{n=1}^{\infty} \frac{\sqrt{n^3-12n-6}}{2n^2-4n-1}$ diverges by the limit comparison test b.c. $\lim_{n\to\infty} \frac{\frac{\sqrt{n^3-12n-6}}{2n^2-4n-1}}{\frac{1}{\sqrt{n}}} = \frac{1}{2} > 0$ And $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges.