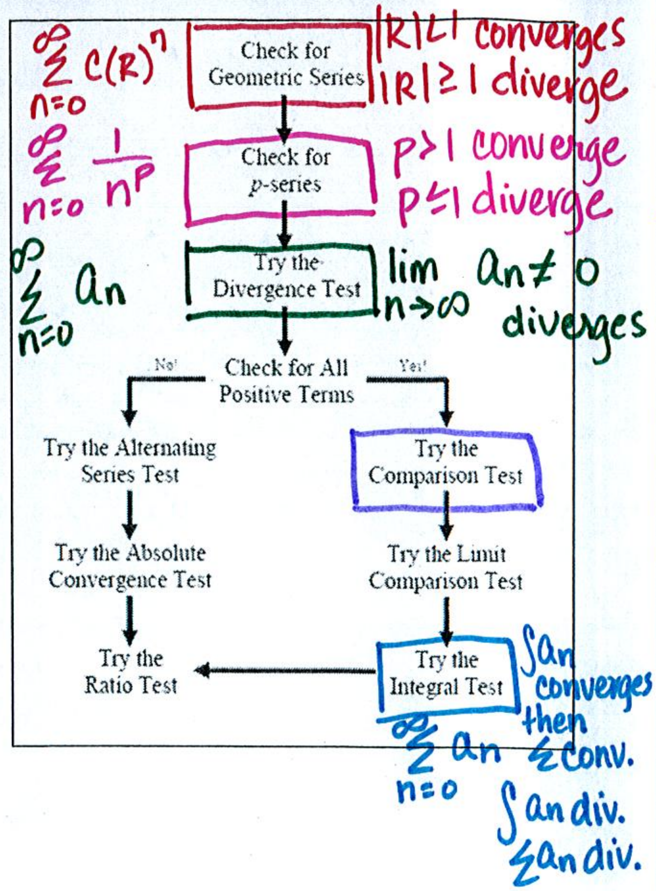
Day 4

Notes: Convergence of Series with Positive Terms




**Integral Test:** Let $a_n = f(n)$, where $f(x)$ is positive, decreasing, and continuous for $x \geq 1$.

$\sum_{n=1}^{\infty} a_n$ converges if $\int_1^{\infty} f(x)dx$ converges.

$\sum_{n=1}^{\infty} a_n$ diverges if $\int_1^{\infty} f(x)dx$ diverges.

Example One: Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. pos, dec, cont. $(1,\infty)$ ✓

$$\int_1^{\infty} \frac{1}{n}\,dn = \lim_{R\to\infty} \int_1^R \frac{1}{n}\,dn$$

$$\lim_{R\to\infty} \ln|n|\Big]_1^R = \lim_{R\to\infty} \ln|R| - \underbrace{\ln|1|}_{0}$$

$$= \lim_{R\to\infty} \ln|R| = \ln|\infty| = \infty$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges b.c. $\frac{1}{n}$ is positive, decreasing, & continuous

And because $\int_1^{\infty} \frac{1}{n}\,dn$ diverges.

Example Two: Determine whether $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges.

$$\int_1^{\infty} \frac{n}{(n^2+1)^2}\,dn = \lim_{R\to\infty} \frac{1}{2}\int_1^R \frac{2n}{(n^2+1)^2}\,dn \quad \begin{matrix} u=n^2+1 \\ du=2n\,dn \end{matrix} = \lim_{R\to\infty} \frac{1}{2}\int_1^R u^{-2}\,du = \lim_{R\to\infty} \frac{1}{2}\frac{u^{-1}}{-1}\Big|_1^R$$

$$\lim_{R\to\infty} \frac{-1}{2(n^2+1)}\Big]_1^R = \lim_{R\to\infty} \frac{-1}{2(R^2+1)} + \frac{1}{2(1^2+1)} = \underbrace{\frac{-1}{2(\infty^2+1)}}_{0} + \frac{1}{2(2)} = \frac{1}{4}$$

$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges b.c. $\frac{n}{(n^2+1)^2}$ is pos, decreasing, & continuous

And b.c. $\int_1^{\infty} \frac{n}{(n^2+1)}\,dn$ converges.

**P-Series:**

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$

Example Three: Determine if each converges or diverges:

A. $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by p-series with $p=1$ which is $\leq 1$.

B. $\sum_{n=1}^{\infty}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by p-series with $p=1$ which is $\leq 1$.

C. $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges by p-series with $p=\frac{3}{2}$ which is $>1$.

D. $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$ diverges by p-series with $p=\frac{1}{2}$ which is $<1$.

**Comparison Test:** Assume that there exists $m > 0$ such that $0 \leq a_n \leq b_n$ for $n \geq m$.

Smaller bigger

$\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges. $a_n < b_n$ & $\sum b_n$ converges

small bigger

$\sum_{n=1}^{\infty} b_n$ diverges if $\sum_{n=1}^{\infty} a_n$ diverges. $a_n < b_n$ & $\sum a_n$ diverges

bigger smaller

Example Four: Show that $\sum_{n=1}^{\infty} 2^{-n!}$ converges $= \sum_{n=1}^{\infty}\frac{1}{2^{n!}}$

$\frac{1}{2^{n!}} \boxed{<} \frac{1}{2^n}$ $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges because geometric $|R|=\frac{1}{2}<1$.

$\sum_{n=1}^{\infty} 2^{-n!}$ converges because $\frac{1}{2^{n!}} < \frac{1}{2^n}$ And $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges.

Example Five: Show that $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}3^n}$ converges.

$\frac{1}{\sqrt{n}\cdot 3^n} \boxed{<} \frac{1}{3^n}$ $\sum_{n=1}^{\infty}\frac{1}{3^n}$ converges b.c. geometric $|R|=\frac{1}{3}<1$.

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\cdot 3^n}$ converges because $\frac{1}{\sqrt{n}\cdot 3^n} < \frac{1}{3^n}$ And $\sum_{n=1}^{\infty}\frac{1}{3^n}$ converges.

Example Six: Determine whether $\sum_{n=2}^{\infty}\frac{1}{\ln(n)}$ converges.

$\frac{1}{\ln(n)} \boxed{>} \frac{1}{n}$ $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges by p-series with $p=1$ which is $\leq 1$.

$\sum_{n=2}^{\infty}\frac{1}{\ln(n)}$ diverges b.c. $\frac{1}{\ln(n)} > \frac{1}{n}$ And $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges.