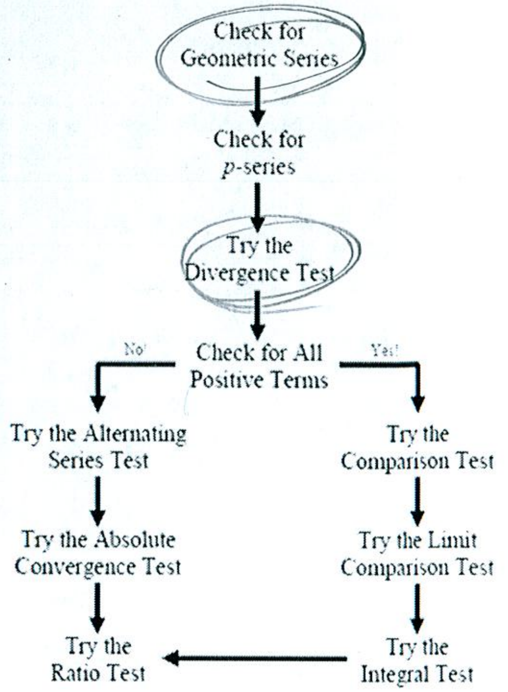
Day 2 & 3

Notes: Geometric, Divergent, & Telescoping Tests



Partial Sums: Given that $S = \sum_{n=1}^{\infty} a_n$,
the nth partial sum is $S_N = \sum_{i=1}^{N} a_i$

Example One: $\sum_{k=1}^{\infty} \frac{1}{k^2}$

Compute the partial sum S_2, S_4, & S_6

$S_2 = \frac{1}{1^2} + \frac{1}{2^2} = \frac{5}{4} = 1.25$

$S_4 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} = \frac{205}{144} \approx 1.424$

$S_6 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} = \frac{5369}{3600} \approx 1.491$

Common Ratio

Geometric Series: Common forms are $\sum_{n=0}^{\infty} cr^n$ and $\sum_{n=1}^{\infty} cr^{n-1}$

You learned to sum these in earlier math classes. This is a review.

Sum of # of terms	$S_N = \sum_{n=1}^{n} cr^n = \frac{c(1-r^n)}{1-r}$	
Sum of all the terms	$S = \sum_{n=1}^{\infty} cr^n = \frac{c}{1-r}$ c – first term $1-r$ – common ratio	To use this formula $\|r\| < 1$. converges if $\|R\| < 1$ diverges if $\|R\| \geq 1$

→ You can find the sum using the formula

Example One: Determine if each converge. If they converge state what they converge to.

A. $\sum_{n=0}^{\infty} 4\left(\frac{1}{5^n}\right) = \sum_{n=0}^{\infty} 4\left(\frac{1}{5}\right)^n$

$S = \frac{\text{first term}}{1 - \text{Ratio}} = \frac{4\left(\frac{1}{5}\right)^0}{\frac{5}{5}1 - \frac{1}{5}} = \frac{4 \cdot \frac{5}{4}}{\frac{4}{5} \cdot \frac{5}{4}} = 5$

$\sum_{n=0}^{\infty} 4\left(\frac{1}{5^n}\right)$ converges because geometric $|R| = \frac{1}{5} < 1$

B. $\sum_{n=3}^{\infty} 2\left(\frac{5}{6}\right)^n$

$S = \frac{\text{first term}}{1 - \text{Ratio}} = \frac{2\left(\frac{5}{6}\right)^3}{\frac{6}{6}1 - \frac{5}{6}} = \frac{\frac{250}{216_{36}} \cdot \frac{6}{1}}{\frac{1}{6} \cdot \frac{6}{1}} = \frac{250}{36} = \frac{125}{18}$

$\sum_{n=3}^{\infty} 2\left(\frac{5}{6}\right)^n$ converges b.c. geometric $|R| = \frac{5}{6} < 1$

C. $\sum_{n=2}^{\infty} 2^{3-2n}$

$\sum_{n=2}^{\infty} 2^3 \cdot 2^{-2n} = \sum_{n=2}^{\infty} 8\left(\frac{1}{2^{2n}}\right) = \sum_{n=2}^{\infty} 8\left(\frac{1}{(2^2)^n}\right) = \sum_{n=2}^{\infty} 8\left(\frac{1}{4^n}\right) = \sum_{n=2}^{\infty} 8\left(\frac{1}{4}\right)^n$

$\sum_{n=2}^{\infty} 2^{3-2n}$ converges b.c. geometric $|R| = \frac{1}{4} < 1$

$S = \frac{8\left(\frac{1}{4}\right)^2}{1 - \frac{1}{4}} = \frac{\frac{1}{2} \cdot \frac{4}{3}}{\frac{3}{4} \cdot \frac{4}{3}} = \frac{2}{3}$

D. $\sum_{n=0}^{\infty} 4(2)^n$

$\sum_{n=0}^{\infty} 4(2)^n$ diverges b.c. geometric $|R| = 2 > 1$

E. $\frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} \ldots\ldots$ $\sum_{n=3}^{\infty} \left(\frac{1^n}{2^n}\right) = \sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^n$ $S = \frac{\left(\frac{1}{2}\right)^3}{1-\frac{1}{2}} = \frac{\frac{1}{8}\cdot\frac{2}{1}}{\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{1}} = \frac{1}{4}$

converges b.c. geometric with $|R| = \frac{1}{2} < 1$

Try These:

1. $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

$5 - \frac{5(2)}{3} + \frac{5(4)}{9} - \frac{5(8)}{27} + \ldots.$

$\sum_{n=0}^{\infty} 5\left(\frac{-2}{3}\right)^n$ converges b.c. geometric

$|R| = \frac{2}{3} < 1$

$S = \frac{5}{1 - -\frac{2}{3}} = \frac{5\cdot\frac{3}{\cancel{5}}}{\frac{\cancel{5}}{3}\cdot\frac{3}{5}} = 3$

2. $\sum_{n=0}^{\infty} \frac{(-3)^{2n}}{10^n} = \sum_{n=0}^{\infty} \frac{\left[(-3)^2\right]^n}{10^n} = \sum_{n=0}^{\infty} \frac{9^n}{10^n}$

$\sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n$ converges because geometric with $|R| = \frac{9}{10} < 1$

3. $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

$\sum_{n=1}^{\infty} (2^2)^n \cdot 3^1 \cdot 3^{-n} = \sum_{n=1}^{\infty} (4)^n \cdot 3\left(\frac{1}{3^n}\right)$

$\sum_{n=1}^{\infty} 3\left(\frac{4}{3}\right)^n$ diverges b.c. geometric with $|R| = \frac{4}{3} > 1$

4. $\sum_{n=2}^{\infty} 3^{2n} 5^{-2n} + 2^{-n}$ $= \frac{1}{2} + \frac{81}{400} = \frac{281}{400}$

$\sum_{n=2}^{\infty} (3^2)^n \left(\frac{1}{(5^2)^n}\right) + \frac{1}{2^n}$

$\sum_{n=2}^{\infty} \left(\frac{9}{25}\right)^n + \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$

converges because geometric with $|R| = \frac{9}{25}$ & $\frac{1}{2} < 1$

$S = \frac{\frac{81}{625}}{1-\frac{9}{25}} = \frac{81}{25\cancel{625}} \cdot \frac{\cancel{25}}{16} = \frac{81}{400}$

$\frac{\frac{1}{4}}{1-\frac{1}{2}}$ $\frac{1}{4}\cdot\frac{2}{1} = \frac{1}{2}$

Divergence Test:

If: $\lim_{n\to\infty} a_n = DNE$ **Then:** $\sum a_n$ *Diverges*

If: $\lim_{n\to\infty} a_n \neq 0$ **Then:** $\sum a_n$ *Diverges*

This test only tells you if $\sum a_n$ diverges

Example Two:

A. $\sum_{3}^{\infty} \frac{3n^4-2n}{7-5n^4}$ $\lim_{n\to\infty} \frac{3n^4-2n}{7-5n^4} = -\frac{3}{5}$

$\sum_{n=3}^{\infty} \frac{3n^4-2n}{7-5n^4}$ diverges b.c $\lim_{n\to\infty} \frac{3n^4-2n}{7-5n^4} = -\frac{3}{5} \neq 0$

B. $\sum_{n=5}^{\infty}\left(\frac{9}{8}\right)^{n}$ $\lim_{n\to\infty}\left(\frac{9}{8}\right)^{n}=\infty$

$\sum_{n=5}^{\infty}\left(\frac{9}{8}\right)^{n}$ diverges because $\lim_{n\to\infty}\left(\frac{9}{8}\right)^{n}=\infty\neq 0$

C. $\sum_{1}^{\infty}\frac{\sqrt[3]{n}}{\ln(n)}$ $\lim_{n\to\infty}\frac{\frac{d}{dn}[n^{1/3}]}{\frac{d}{dn}[\ln(n)]}=\lim_{n\to\infty}\frac{\frac{1}{3}n^{-2/3}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\frac{1}{3n^{2/3}}\cdot\frac{n}{1}}{\frac{1}{n}\cdot\frac{n}{1}}=\lim_{n\to\infty}\frac{n}{3n^{2/3}}=+\infty$

$\sum_{n=1}^{\infty}\frac{\sqrt[3]{n}}{\ln(n)}$ diverges b.c. $\lim_{n\to\infty}\frac{n^{1/3}}{\ln(n)}=+\infty\neq 0$

Telescoping/Harmonic Series:

Example Three: Find the sum of the following convergent series:

A. $\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1}=\boxed{1}$

$$\overset{n=1}{\left(1-\cancel{\tfrac{1}{2}}\right)}+\overset{n=2}{\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)}+\overset{n=3}{\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}\right)}+\overset{n=4}{\left(\cancel{\tfrac{1}{4}}-\cancel{\tfrac{1}{5}}\right)}+\overset{n=5}{\left(\cancel{\tfrac{1}{5}}-\cancel{\tfrac{1}{6}}\right)}+\overset{n=6}{\left(\cancel{\tfrac{1}{6}}-\cancel{\tfrac{1}{7}}\right)}+\ldots+\overset{n=\infty}{\left(\cancel{\tfrac{1}{\infty}}-\underset{\underset{0}{\downarrow}}{\tfrac{1}{\infty}}\right)}$$

B. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ $\boxed{\text{let } n=-1}$ $\boxed{\text{let } n=0}$ $\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1}=\boxed{1}$

$\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}$

$1=-1B$ $1=1A$

$B=-1$ $A=1$

$1=A(n+1)+B(n)$

~~Already answered this.~~ Y'all Suck!

C. $\sum_{n=1}^{\infty}\frac{5}{n(n+2)}$ $\frac{5}{n(n+2)}=\frac{A}{n}+\frac{B}{n+2}$ $5=A(n+2)+B(n)$

$\boxed{\text{let } n=-2}$ $\boxed{\text{let } n=0}$

$5=-2B$ $5=2A$

$B=-\frac{5}{2}$ $A=\frac{5}{2}$

$\sum_{n=1}^{\infty}\frac{5/2}{n}-\frac{5/2}{n+2}$

$\frac{5}{2}\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+2}$

$$\frac{5}{2}\left[\left(1-\cancel{\tfrac{1}{3}}\right)+\left(\tfrac{1}{2}-\cancel{\tfrac{1}{4}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{5}}\right)+\left(\cancel{\tfrac{1}{4}}-\cancel{\tfrac{1}{6}}\right)+\left(\cancel{\tfrac{1}{5}}-\cancel{\tfrac{1}{7}}\right)+\ldots\left(\cancel{\tfrac{1}{\infty}}-\cancel{\tfrac{1}{\infty+2}}\right)\right]$$

$\frac{5}{2}\left[1+\frac{1}{2}\right]=\frac{5}{2}\left[\frac{3}{2}\right]=\frac{15}{4}$