Name________________________ Date__________

Worksheet: Taylor Series from AP BC Free Response Questions

1995-BC 6 (No Calculator)

1. Let f be a function that has derivatives of all orders for all real numbers. Assume f(1) = 3 , f'(1) = -2, $f''(\$) = 2$, and $f'''(\$) = 4$.

a. Write the second-degree Taylor polynomial for f(x) about x = 1 and use it to approximate f(0.7).

$T_2(x) = 3 - 2(x-1) + (x-1)^2$

$T_2(.7) \approx 3.69$

b. Write the third-degree Taylor polynomial for f(x) about x = 1 and use it to approximate f(1.2).

$T_3(x) = 3 - 2(x-1) + (x-1)^2 + \frac{2}{3}(x-1)^3$

$T_3(1.2) \approx 2.645$

c. Write the second degree Taylor polynomial for f'(x) about x = 1 and use it to approximate f'(1.2).

$f'(x) = -2 + 2(x-1) + 2(x-1)^2$

$f'(1.2) \approx -1.52$

2000- BC 3 (Calculator)

2. The Taylor series about x = 5 for a certain function converges to f(x) for all x in the interval of convergence. The nth derivative of f(x) at x = 5 is given by $f^{(n)}(5) = \frac{(-1)^n n!}{2^n(n+2)}$, and $f(5) = \frac{1}{2}$.

a. Write the third-degree Taylor polynomial for f(x) about x = 5.

$f(5) = \frac{1}{2}$

$f'(5) = \frac{(-1)^1 1!}{2(1+2)} = \frac{-1}{6}$

$f''(5) = \frac{(-1)^2 2!}{2^2(2+2)} = \frac{2!}{4(4)} = \frac{1}{8}$

$f'''(5) = \frac{(-1)^3 3!}{2^3(3+2)} = \frac{-3!}{8 \cdot 5} = \frac{-6}{40}$

$T_3(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{8}\frac{(x-5)^2}{2} - \frac{6}{40}\frac{(x-5)^3}{3!}$

$\boxed{T_3(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3}$

b. Find the radius of convergence of the Taylor series representation for f(x) about x = 5.

$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{2^n(n+2)} \cdot \frac{(x-5)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^n}{2^n(n+2)}$

$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)}}{\frac{(-1)^n(x-5)^n}{2^n(n+2)}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(x-5)^n}(x-5)}{\cancel{2^n} \cdot 2(n+3)} \cdot \frac{\cancel{2^n}(n+2)}{\cancel{(x-5)^n}} \right| = \lim_{n\to\infty} \frac{n+2}{2n+6}|x-5| = \frac{1}{2}|x-5| < 1$

$|x-5| < 2$

$\therefore \boxed{R = 2}$

c. Show that the sixth-degree Taylor polynomial for f about x = 5 approximates f(6) with error less than $\frac{1}{1000}$.

$|f(6) - T_6(6)| \leq \frac{k}{(n+1)!}|x-a|^{n+1}$

$\frac{k}{(n+1)!}|x-a|^{n+1}$

$\frac{k}{(6+1)!}(5-6)^{6+1}$

$k: \left|f^{(7)}(5)\right| = \left|\frac{(-1)^7 7!}{2^7(7+2)}\right| = |f^{(7)}(6)|$ (both the same value)

$\therefore k = \frac{7!}{2^7(9)}$

$\frac{\frac{7!}{2^7(9)}}{7!}(-1)^6 = \frac{1}{2^7(9)} = \frac{1}{1152}$

Since $\frac{1}{1152} < \frac{1}{1000}$,

then $T_6(x)$ approximates with error less than $\frac{1}{1000}$

1998-BC 3 (Calculator)

3. Let f be a function that has derivatives of all orders for all real numbers. Assume f(0) = 5 , f'(0) = -3, f''(0) = 1, and f'''(0) = 4.

a. Write the third-degree Taylor polynomial for f(x) about x = 0 and use it to approximate f(0.2).

$$T_3(x) = 5 - 3x + \frac{x^2}{2} + \frac{4x^3}{3!}$$

$$T_3(.2) \approx 4.425$$

b. Write the fourth-degree Taylor polynomial for g(x) where g(x) = f(x^2), about x = 0.

$$T_2(x^2) = 5 - 3x^2 + \frac{x^4}{2}$$

sub x^2 in for x

c. Write the third-degree Taylor polynomial for h(x) where h(x) = $\int_0^x f(t)dt$, about x = 0.

$$h(x) = \int_0^x 5 - 3t + \frac{t^2}{2}\,dt \qquad h(x) = 5x - \frac{3x^2}{2} + \frac{x^3}{6}$$

* term by term antidifferentiation

2002 (No Calculator)

4. The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \ldots + \frac{(2x)^{n+1}}{n+1} + \ldots$$

on its interval of convergence.

a. Find the interval of convergence of the Maclaurin series for f. Justify your answer.

$$\lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+2}}{n+2}}{\frac{(2x)^{n+1}}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(2x)^n(2x)^2}{n+2} \cdot \frac{n+1}{(2x)^n(2x)^1} \right|$$

$$= \lim_{n\to\infty} \frac{2(n+1)}{n+2}|x| = \lim_{n\to\infty} 2|x| < 1$$

$$|x| < \frac{1}{2} \qquad \boxed{-\frac{1}{2} < x < \frac{1}{2}} \qquad \left[-\frac{1}{2}, \frac{1}{2}\right)$$

$x = \frac{1}{2}$ $\sum \frac{(1)^{n+1}}{n+1}$ Diverges comparison test

Diverges p-series p≤1

compare to $\sum \frac{1}{n}$

$x = -\frac{1}{2}$ $\sum \frac{(-1)^{n+1}}{n+1}$ Converges alt. series test

b. Find the first four terms and the general term for the Maclaurin series for f'(x) .

$$f'(x) = 2 + 4x + 8x^2 + 16x^3 + \ldots + 2(2x)^n$$

$$\sum_{n=0}^{\infty} 2(2x)^n$$

c. Use the Maclaurin series you found in part B to find the value of f'(-1/3).

$$f'(-\tfrac{1}{3}) = 2 + 4(-\tfrac{1}{3}) + 8(-\tfrac{1}{3})^2 + 16(-\tfrac{1}{3})^3 + \ldots + 2(2(-\tfrac{1}{3}))^n$$

$\sum 2(-\frac{2}{3})^n$ ← geometric series!

$$S_n = \frac{2}{1-(-\frac{2}{3})} = \frac{2}{1+\frac{2}{3}} = \frac{2}{\frac{5}{3}} = \boxed{\frac{6}{5}}$$

2015 (No Calculator)

5. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - ... + \frac{(-3)^{n-1}}{n}x^n + ...$ and converges to f(x) for |x| < R, where R is the radius of convergence of the Maclaurin series.

a. Use the ratio test to find R.

$$\lim_{n\to\infty} \left| \frac{\frac{(-3)^n x^{n+1}}{n+1}}{\frac{(-3)^{n-1}x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{3^n x^n \cdot x}{n+1} \cdot \frac{n}{3^n 3^{-1} x^n} \right| = \lim_{n\to\infty} \frac{3n}{n+1}|x| = 3|x| < 1$$

$$|x| < \frac{1}{3}$$

$$\boxed{R = \frac{1}{3}}$$

b. Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for |x| < R.

$$f'(x) = 1 - 3x + 9x^2 - 27x^3 = 1 - (3x) + (3x)^2 - (3x)^3$$

looks like $\frac{1}{1+x}$:)

$$\therefore f'(x) \ \frac{1}{1+3x}$$

c. Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about x = 0

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

$$e^x f(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3}\right)\left(x - \frac{3}{2}x^2 + 3x^3\right) = x - \frac{1}{2}x^2 + 2x^3$$

2007 Form B (No Calculator)

6. Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

a. Find the first four nonzero terms and the general term for the Taylor series for f about x = 0.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \quad \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x/3} = 1 - \frac{x}{3} + \frac{x^2}{18} - \frac{x^3}{27 \cdot 3!} + \cdots$$

$$6e^{-x/3} = 6 - 2x + \frac{x^2}{3} - \frac{x^3}{27} + \cdots$$

$$\sum_{n=0}^{\infty} \frac{6\left(-\frac{x}{3}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 6x^n}{3^n n!}$$

b. Let g be the function given by $g(x) = \int_0^x f(t)dt$. Find the first four nonzero terms and the general term for the Taylor series for g about x = 0.

$$\int_0^x f(t)dt = \int_0^x 6 - 2t + \frac{t^2}{3} - \frac{t^3}{27} + \cdots$$

$$= 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{108} + \cdots \quad \sum_{n=0}^{\infty} \frac{(-1)^n 6x^{n+1}}{3^n (n+1)n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 6x^{n+1}}{3^n (n+1)!}$$