AP Calculus
Logistic Equation & Euler's Method

Name________________________________
Date____________________ Period__________

1. Given that $y(1) = -3$ and $\frac{dy}{dx} = 2x + y$, what is the approximation for $y(2)$ if Euler's method is used with a step size of 0.5, starting at $x = 1$?

(A) -5
(B) -4.25
(C) -4
(D) -3.75
(E) -3.5

x	$(2x+y)(.5)$	y
1		-3
1.5	$[(2)(1)+(-3)](\frac{1}{2}) = -\frac{1}{2}$	$-3-.5=-3.5$
2	$[(2)(1.5)+(-3.5)]\frac{1}{2} = -\frac{1}{2}(\frac{1}{2}) = -\frac{1}{4}$	$-3.5-.25=-3.75$

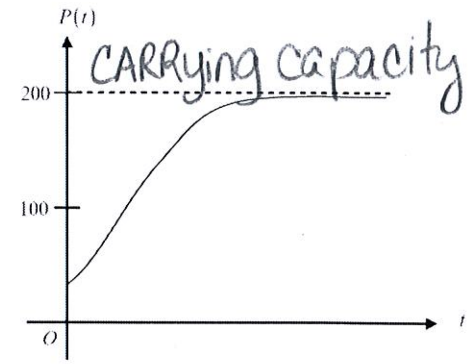
2. Which of the following differential equations for a population P could model the logistic growth shown in the figure?

(A) $\frac{dP}{dt} = 0.2P - 0.001P^2$
(B) $\frac{dP}{dt} = 0.1P - 0.001P^2$
(C) $\frac{dP}{dt} = 0.2P^2 - 0.001P$
(D) $\frac{dP}{dt} = 0.1P^2 - 0.001P$
(E) $\frac{dP}{dt} = 0.1P^2 + 0.001P$

$\frac{dP}{dt} = kP\left[1 - \frac{P}{A}\right]$

$\boxed{A}$ $.2P[1 - .005P]$

$.2P\left[1 - \frac{P}{200}\right]$



CARRying capacity 200

3. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x + y$ with the initial condition $f(1) = 2$. What is the approximation for $f(2)$ if Euler's method is used, starting at $x = 1$ with a step size of 0.5?

(A) 3
(B) 5
(C) 6
(D) 10
(E) 12

x	$(x+y)(\frac{1}{2})$	y
1		2
1.5	$(1+2)\frac{1}{2} = \frac{3}{2} = 1.5$	$2+1.5=3.5$
2	$(1.5+3.5)\frac{1}{2} = \frac{5}{2} = 2.5$	$3.5+2.5=6$

4. The number of moose in a national park is modeled by the function M that satisfies the logistic differential equation $\frac{dM}{dt} = 0.6M\left(1 - \frac{M}{200}\right)$, where t is the time in years and $M(0) = 50$. What is $\lim_{t\to\infty} M(t)$?

(A) 50
(B) 200
(C) 500
(D) 1000
(E) 2000

$\frac{dM}{dt} = kM\left(1 - \frac{M}{A}\right)$

200 = caRRying capacity

limit = caRRying capacity

5. The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where the initial population $P(0) = 3{,}000$ and t is the time in years. What is $\lim_{t\to\infty} P(t)$?

(A) 2,500
(B) 3,000
(C) 4,200
(D) 5,000
(E) 10,000

$\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$

$\frac{dP}{dt} = 2P\left[1 - \frac{P}{2(5000)}\right]$

$\frac{dP}{dt} = 2P\left[1 - \frac{P}{10{,}000}\right]$

limit = caRRing capacity

limit = 10,000

AP Calculus
2005 Problem 4

Name__________________
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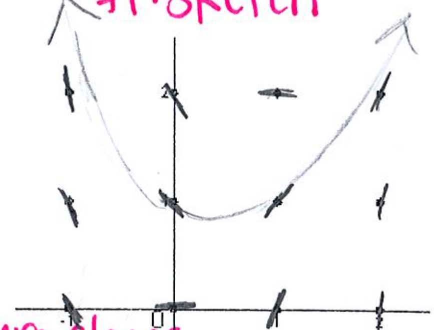
Consider the differential equation $\frac{dy}{dx} = 2x - y$.

a.) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(0,1)$

+1: sketch



$(-1,0)=-2 \quad (0,0)=0 \quad (1,0)=2 \quad (2,0)=4$

$(-1,1)=-3 \quad (0,1)=-1 \quad (1,1)=1 \quad (2,1)=3$

$(-1,2)=-4 \quad (0,2)=-2 \quad (1,2)=0 \quad (2,2)=2$

+1: zero slopes
+1: non zero slopes

b.) The solution curve that passes through the point $(0,1)$ has a $\underbrace{\text{local minimum at}}_{\frac{dy}{dx}=0}$ $x = \ln\left(\frac{3}{2}\right)$.

What is the y-coordinate of this local minimum?

$0 = 2x - y$ +1

$0 = 2\ln\left(\frac{3}{2}\right) - y \quad \boxed{y = 2\ln(3/2)}$ +1

c.) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size to approximate $f(-0.4)$. Show the work that leads to your answer.

0	$(2x-y)(-.2)$	1
$-.2$	$[2(0)-1](-.2)$ $(-1)(-.2)=(.2)$	1.2
$-.4$	$[2(-.2)-1.2](.2)$ $[-.4-1.2](.2) = +.32$	1.52 +1

-1.6
$\times .2$
$.32$

+1: Process

d.) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.

$\frac{d}{dx}\left[\frac{dy}{dx} = 2x - y\right]$

$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$

$\frac{d^2y}{dx^2} = 2 - [2x - y]$

$\frac{d^2y}{dx^2} = 2 - 2x + y$ +1

$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = 2 - 2(0) + 1 = 3$

+1 $\left\{ \begin{array}{l} \frac{d^2y}{dx^2} \text{ at } (0,1) \text{ is positive} \therefore \\ f(x) \text{ is concave up so} \\ \text{the approximation is an} \\ \text{underapproximation.} \end{array} \right.$

AP Calculus
2004 Problem 5

Name____________________
Date______________Period___

A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

a.) If $P(0) = 3$, what is $\lim_{t\to\infty} P(t)$? $= 12$ +1

If $P(0) = 20$, what is $\lim_{t\to\infty} P(t)$? $= 12$ +1

Initial condition does not matter.

$\lim_{t\to\infty} P(t) =$ carrying capacity

b.) If $P(0) = 3$, for what value of P is the population growing fastest?

$P = 6$ +1

Growing fastest at half Carrying Capacity

c.) A differential population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

t Y

$\frac{dY}{dt} = \frac{1}{5}Y\left(1 - \frac{t}{12}\right)$

+1 $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{1}{12}t\right)dt$

+1 $\ln|Y| = \frac{1}{5}\left(t - \frac{1}{12}\frac{t^2}{2}\right) + C$ +1

$\ln(3) = \frac{1}{5}\left[0 - \frac{1}{24}(0)^2\right] + C$ +1

$C = \ln(3)$

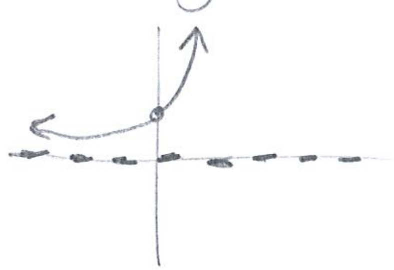
$\ln|Y| = \frac{1}{5}t - \frac{1}{120}t^2 + \ln 3$

$e \quad e$

$Y = 3e^{\frac{1}{5}t - \frac{1}{120}t^2}$ +1

d.) For the function Y found in part (c), what is $\lim_{t\to\infty} Y(t)$?

Parent: $y = e^x$



$\lim_{t\to\infty} Y(t) = 0$ +1

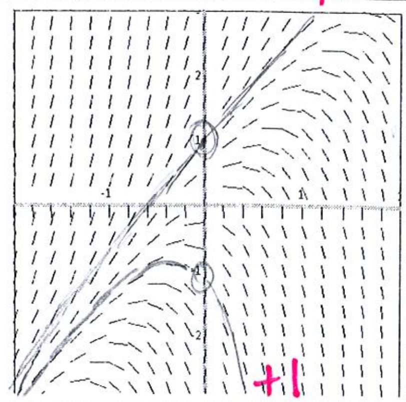
AP Calculus
2002 Problem 5

Name____________________
Date______________Period___

Consider the differential equation $\frac{dy}{dx} = 2y - 4x$

a.) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point (0,1) and sketch the solution curve that passes through the point (0,-1).

+1



+1

b.) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1 to approximate $f(.2)$. Show the work that leads to your answer.

0	$[2y-4x](.1)$	1
.1	$[2(1)-4(0)](.1)$ $\underbrace{[2](.1) = .2}_{+1\text{:method}}$	1.2
.2	$[2(1.2)-4(.1)](.1)$ $[2.4-.4](.1) = .2$	1.4 +1

c.) Find the value of b for which $y = \boxed{2x + b}$ is a solution to the given differential equation. Justify your answer.

$m=2$

$\frac{dy}{dx} = 2\boxed{y} - 4x$

$\frac{dy}{dx} = 2(2x+b) - 4x$

$2 = \cancel{4x} + 2b - \cancel{4x}$

+1

$2 = 2b \quad b = 1$ +1

d.) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0,0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

$\left.\frac{dy}{dx}\right|_{(0,0)} = 2(0) - 4(0) = 0 \therefore$ critical pt. +1

$\frac{d}{dx}\left[\frac{dy}{dx} = 2y - 4x\right] = \frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 4 = 2[2y-4x] - 4 = 4y - 8x - 4$

$\left.\frac{d^2y}{dx^2}\right|_{(0,0)} = 4(0) - 8(0) - 4 = -4$ +1

$g'(0) = 0$

+1 $\therefore g(0) = 0$ is a max bc $g''(0) = $ neg

AP Calculus
2001 Problem 5

Name____________________
Date______________Period___

Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$ and $\lim_{x\to\infty} f(x) = 0$.

a.) Evaluate $\int_1^\infty -3xf(x)dx$. Show the work that leads to your answer.

$$\lim_{R\to\infty} \int_1^R -3xf(x)\,dx \qquad f'(x) = \boxed{-3xf(x)}$$

$$\lim_{R\to\infty} \int_1^R f'(x)\,dx = \underline{\lim_{R\to\infty} f(x)\Big]_1^R}\ +1 = \lim_{R\to\infty} f(R) - f(1)$$

$$0 - 4 = \underline{-4}\ +1$$

b.) Use Euler's method, starting at $x = 1$ with a step size of 0.5, to approximate $f(2)$.

1	$-3xf(x)\cdot(.5)$	4
1.5	$-3(1)f(1)[.5]$ $\underbrace{-3(4)(.5) = -6}_{+1:\text{method}}$	-2
2	$-3(1.5)f(1.5)(.5)$ $-4.5(-2)(.5) = 4.5$	$\underline{2.5}$ +1

c.) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = -3xy$ with the initial condition $f(1) = 4$.

$$\frac{dy}{dx} = -3xy$$

$$\underline{\frac{1}{y}dy = -3x\,dx}\ +1$$

$$\int \frac{1}{y}dy = \int -3x\,dx$$

$$+1\ \underline{\ln|y| = -\frac{3x^2}{2}}\ \underline{+C}\ +1$$

$$\underline{\ln|4| = -\frac{3}{2}(1)^2 + C}\ +1$$

$$C = \ln(4) + \frac{3}{2}$$

x y

$$\ln|y| = -\frac{3}{2}x^2 + \ln 4 + \frac{3}{2}$$

$$e^{\ln|y|} = e^{-\frac{3}{2}x^2 + \frac{3}{2} + \ln 4}$$

$$|y| = 4e^{-\frac{3}{2}x^2 + \frac{3}{2}}$$

$$y = \pm 4e^{-\frac{3}{2}x^2 + \frac{3}{2}}$$

$$4 = \pm 4e^{-\frac{3}{2}(1)^2 + \frac{3}{2}}$$

$$\underline{y = 4e^{-\frac{3}{2}x^2 + \frac{3}{2}}}\ +1$$

AP Calculus
2000 Problem 6

$x=0$
$m=0$

Name____________________
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Consider the differential equation given by $\frac{dy}{dx} = x(y-1)^2$

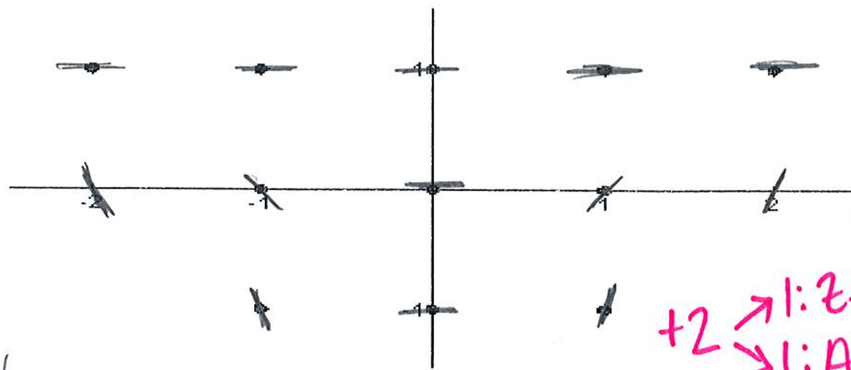
$y=1$
$m=0$

a.) On the axis provided, sketch a slope field for the given differential equation at the eleven points indicated.

$(-2,0) = (-2)[0-1]^2 = -2$

$(-1,0) = (-1)[0-1]^2 = -1$

$(-1,-1) = (-1)[-1-1]^2 = -4$



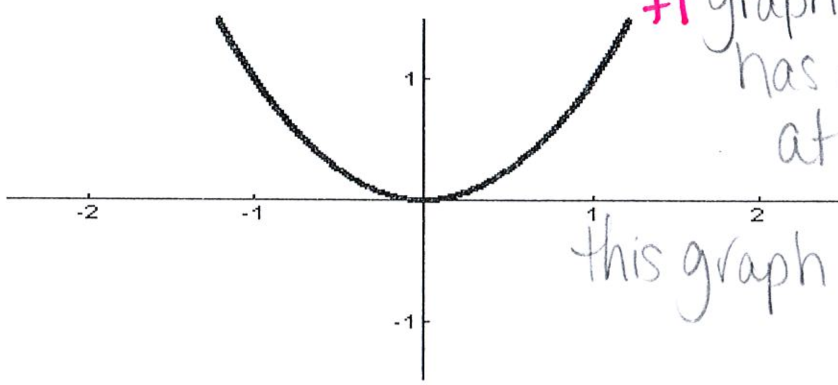
$(2,0) = 2[0-1]^2 = 2$

$(1,0) = 1(0-1)^2 = 1$

$(1,-1) = 1[-1-1]^2 = 4$

+2 → 1: Zero slopes
→ 1: All other

b.) Use the slope field for the given differential equation to explain why a solution could not have the graph shown below.



+1 Graph above has a slope = 0 at $y=1$
this graph does not

c.) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -1$.

$\frac{dy}{dx} = x(y-1)^2$

$\frac{1}{(y-1)^2}dy = x\,dx$ +1

$u = y-1$
$du = dy$

$\int \frac{1}{u^2}du = \int x\,dx$

$\int u^{-2}du = \int x\,dx$

$\frac{u^{-1}}{-1} = \frac{1}{2}x^2 + C$ +1 +1

$\frac{-1}{y-1} = \frac{1}{2}x^2 + C$

$\frac{-1}{-1-1} = \frac{1}{2}(0)^2 + C$ +1

$\frac{1}{2} = C$

$\frac{-1}{y-1} = \frac{1}{2}x^2 + \frac{1}{2}$

$y-1 = \frac{(-1)2}{(\frac{1}{2}x^2+\frac{1}{2})2}$

$y = \frac{-2}{x^2+1} + 1$

+1

d.) Find the range of the solution found in part (c).

$y = \frac{-2}{x^2+1} + \frac{1(x^2+1)}{(x^2+1)} = \frac{x^2+1-2}{x^2+1} = \frac{x^2-1}{x^2+1}$

$y = \frac{x^2-1}{x^2+1}$

$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$ largest y-value

$\lim_{x\to 0} \frac{x^2-1}{x^2+1} = -1$ smallest y-value

$[-1, 1]$ +1